In memoriam Aretin Corciovei

POISSON-LIE STRUCTURES AND QUANTISATION WITH CONSTRAINTS

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1. INTRODUCTION

The quantisation of systems with constraints is old as the quantum mechanics itself. The first such problem brilliantly solved was the finding of the hydrogen atom spectrum by Pauli in 1926 [1]. Enforcing the constraints in classical mechanics has a satisfactory solution [2, 3], but this is no more true in quantum mechanics. The constraints, i.e. a set of functions

\[ \phi_i(q, p) = 0, \quad i = 1, 2, \ldots, m \] (1.1)

restrict the motion of the classical system to a manifold embedded in the initial Euclidean phase space and in consequence the canonical quantisation rules

\[ \{q_i, p_j\} = i\hbar \delta_{ij} \]

are no more sufficient for the quantum description of the physical system.

In general quantisation is not a well-defined procedure existing today a variety of methods which sometimes give different results when applied to physical problems, although the starting points are similar from the classical point of view. We mention only the people who derive the Schrödinger equation by

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Feynman's path integral method; see for example [4–6], who find an extra energy term proportional to the Riemann scalar curvature of the manifold, even for the simple case of the motion of a particle on the n-dimensional sphere.

The most successful method for imposing quantum constraints is that found by Dirac [7], however nowadays there are some voices who reject it claiming that the resulting energy spectrum is incorrect even for simple systems [8, 9]. The mechanism found by Dirac was the introduction of a new symplectic structure, the Dirac bracket, to handle the second-class constraints and the use of the Legendre multipliers to find the true Hamiltonian.

The purpose of this paper is to look at the problem of quantisation with constraints from a slightly modified point of view and to show that the new proposal leads to correct results.

When one studies constrained systems one starts with a Hamiltonian, $H(q, p)$, and a number of relations of the form (1.1), called primary constraints. In the new situation the "unexpected" thing which appears is that the "Hamiltonian", which is considered a basic observable in classical as well as quantum physics, does not commute with the constraints, i.e.

$$\{\phi_i, H\} = \frac{d\phi_i}{dt} = 0$$

(1.2)

Thus the constraints do not conserve in time which suggests that the Hamiltonian one starts with is not the true one and a solution for solving this dilemma was that found by Dirac [7]. In a not too much different situation, that concerning the description of a spinning particle, the problem has been treated from the beginning in quantum terms and the non-commutativity of the operators was accepted as being normal since no classical analog of spin was known at that time. Although the situation appearing in (1.2) is not essentially different from the preceding one the emotional value of the energy notion, the Hamiltonian, has impeded the finding of the natural solution, since if the energy does not exist (as an observable), nothing exists! We consider that the natural solution could be that by looking at the Hamiltonian and the constraints as components or generators of a larger symmetry. In this case the classical Hamiltonian is one of the many other generators and by consequence it has no more "rights" than its fellows entering the game. If the problem makes sense the observable(s) will be given by the Casimir(s) of the new algebra and, of course, there will exist a common base of eigenvectors for the Casimir(s) and the old Hamiltonian.

In the most simple cases the Hamiltonian and the constraints such as (1.1) by Poisson bracket technique generate a Poisson algebra of the form

$$\{H(q, p), \phi_j(q, p)\} = C^j_i \phi_i(q, p)$$

$$\{\phi_i(q, p), \phi_j(q, p)\} = C^k_i \phi_k(q, p)$$

(1.3)
where $C^i_j$ and $C^i_k$ are constant structure coefficients. In our opinion this Poisson structure is the basic structure for the quantisation procedure. Since the Poisson algebra (1.3) transforms by quantisation into a Lie algebra the physical observables of the model will be given as we said before by the Casimir operators; this means that in general no one of the initial operators transform into a veritable observable, this being possible only for those generators that are in the center of the algebra, i.e. only for non semisimple Lie algebras.

We applied this idea by using an ad hoc procedure to the motion of a particle on the $n$-dimensional sphere $S^n$ and we have found that the "Hamiltonian", i.e. the Casimir of the corresponding algebra is a quadratic function in the old Hamiltonian and the constraints [10].

In this paper we want to generalize this procedure by looking for other simple physical systems which can be solved by our formalism. The idea is to look at cases which generate more general Poisson-Lie algebras. These objects are nowadays well known [11] and their study has began with the quadratic Poisson algebras introduced by Sklyanin [12]. In our opinion the first quadratic algebra entering quantum physics was that found by Pauli when solving the eigenvalue problem for the hydrogen atom [1]. In that case the constraints are those quantities which make the classical orbit to be closed and are given by the components of the Laplace-Runge-Lenz vector. Denoting them as usual by $M_i$, the Poisson brackets have the form

$$\{M_i, M_j\} = -\frac{2}{m} \epsilon_{ijk} HL_k \quad i, j = 1, 2, 3$$

where $H$ is the Hamiltonian and $L_i$ are the components of the angular momentum. First of all the above algebra is quadratic as the preceding relations show. On the other hand the Hamiltonian entering the above relation commutes with $L_i$ and $M_i$, the algebra is not semisimple and the classical Hamiltonian is one of the Casimir operators of the algebra, i.e. it is a conserved quantity, but this information is not enough for finding its discret spectrum. In our opinion these were the main facts which impeded the recognition of the new structure and of its predicting power. In that case, as we shall see, the spectrum is given by another Casimir of the algebra which is a linear combination between $L^2$ and $M^2$. The spectrum of this Casimir depends parametrically on $H$, which is a good quantum number, and by solving the spectrum equation with respect to $H$ one gets the known discrete spectrum of the hydrogen atom.

Non semisimple algebras have appeared recently in the construction of Wess-Zumino-Witten conformal field theory models [13] and they are nowadays accepted as models for physical processes.

In Section 2 we present the mathematical formalism and in Section 3 we treat the motion of a particle on an $n$-dimensional sphere and the hydrogen atom
from the point of view of our formalism. In the next section we will illustrate the method by other examples of Poisson-Lie structures which appear in other physical problems. The paper ends with Conclusion.

2. MATHEMATICAL FORMALISM

In this section we want to extend our method which was applied to a particular case in [10] to more general situations than those given by Eqs. (1.3) by developing a formalism which makes use of the Lie algebra properties of the Poisson bracket. We hope that this formalism will solve at least a part of the problems encountered in quantisation with constraints.

More precisely let \((u_1, u_2, \ldots, u_r)\) denote \(r\) functions of \(2n\) independent variables \((q_1, \ldots, q_n, p_1, \ldots, p_n)\) and suppose that all Poisson brackets \(\{u_i, u_j\}\) can be expressed as functions of \((u_1, u_2, \ldots, u_r)\). In this case these functions form a Poisson-Lie structure and any function of \((u_1, u_2, \ldots, u_r)\) belongs to this algebra. This kind of structure was first introduced by S. Lie who used the name of function group [14]. The full phase space is \(\mathbb{R}^{2n}\) with generic point \((q, p)\) and the usual Poisson algebra \(\mathcal{P} = (C^\infty(\mathbb{R}^{2n}), [, ])\) is the setting for the problem.

The systems with constraints are good candidates for such structures since we start with a Hamiltonian and a number of primary constraints of the form (1.1). By taking the Poisson brackets \(\{H, q_i\}\) and \(\{q_i, p_j\}\) one generates secondary constraints. Let suppose that this process closes and at the end we obtain a finite number of independent dynamical variables \((u_1, \ldots, u_r)\) which describe the dynamics of the constrained system. These dynamical variables satisfy a system of equations of the following form

\[
\{u_i, u_j\} = f_{ij}(u_1, \ldots, u_r) \quad i, j = 1, 2, \ldots, r, \quad i < j
\]

with \(f_{ij} = -f_{ji}\) skew-symmetric functions. If \(f_{ij}\) has a power series expansion this may have the form

\[
f_{ij}(u_1, \ldots, u_r) = a_{ij} + b_{ij}u_k + c_{ij}u_ku_l + \ldots
\]

In this approach we make no distinction between Hamiltonian, primary and secondary constraints, first or second class constraints, all of them are simply dynamical variables living in a democratic society, the rules which they obey being the system of equations (2.1). Because in the new situation the Hamiltonian is only one of the dynamical variables having no privileged status we have to solve the problem of integrals of motion for a dynamical system governed by
Eqs. (2.1). It seems natural to extend the classical solution, $F$ is an integral of motion if $\{F, H\} = 0$, to the new context by requiring that $F$ is an integral of motion if

$$\{F, u_i\} = 0, \quad i = 1, 2, \ldots, r$$

i.e. the integral of motion is that quantity which commutes with all the dynamical variables: the constraints and the Hamiltonian.

We remind that the same condition was used by Dirac too[7], but only in the new simplectic structure, the Dirac bracket, $\{\cdot, \cdot\}_D$, and not in the canonical Poisson structure as we do here. Taking into account the Poisson-Lie structure defined by Eqs. (2.1) the above equations are equivalent to the following system of first order partial differential equations

$$\sum_{i=1}^{ir} \frac{\partial F}{\partial u_i} u_j = \sum_{i=1}^{ir} \frac{\partial F}{\partial u_i} f_{ij}(u_1, \ldots, u_r) = 0$$

(2.2)

$$j = 1, \ldots, r$$

Being a homogeneous system a necessary condition for the existence of a non-trivial solution, $F \neq ct$, is [15, 16]

$$\det | [u_i, u_j]| = \det | f_{ij}(u_1, \ldots, u_r) | = 0$$

(2.3)

One can easily construct Hamiltonians and constraints for which Eq. (2.2) is satisfied, but the system of equations has no solutions. This may happen when some important physical information was not taken properly into account, but there could be cases when the stated problem has no physical meaning.

The solution(s) of the system (2.2) will depend in general on all dynamical variables and will play the rôle played by the classical Hamiltonian for non constrained systems, they being the conserved physical quantities of the dynamical system. The classical theory of first order partial differential equations tells us that if the rank of the system (2.1) is $\rho$ then (2.1) may have up to $n = r - \rho$ independent solutions and the easiest way to obtain them is by using the characteristic method [15, 16]. The simplest solutions of the system (2.1) are called elementary solutions, the general solution being an arbitrary continuous and derivable function of these elementary solutions $G = G(F_1, F_2, \ldots)$. By quantization $\{\cdot, \cdot\}$ goes into $\frac{1}{i\hbar} [\cdot, \cdot]$ and the observables of the theory will be the solutions of the system (2.2). When the algebra (2.1) reduces to that of a semisimple Lie algebra the solutions $F_i$ will be the Casimir operators of this algebra and if the respective algebra has rank $\ell$ there will be $\ell$ Casimir operators by the well-known result by Racah [17]. Thus Eqs. (2.1)–(2.2) represent a generalization of the known powerful machinery of representation theory of Lie algebras and give us
a method for finding the maximal set of commuting observables for a given physical system. Finding the physically relevant operators and their spectra is one of the goals of any quantum theory.

3. APPLICATIONS

In the following we illustrate the new method by considering again the quantisation of the motion of a particle on the sphere and the hydrogen atom. We consider first the motion of a particle on the \( n \)-dimensional sphere which is the toy model for testing quantum constrained dynamics [8, 10, 18, 19]. The free Hamiltonian is

\[
H = \frac{1}{2} (p, p) = \frac{p^2}{2},
\]

where \((p, p)\) denotes the Euclidean scalar product in the \( n + 1 \)-dimensional space, i.e. \( \sum_{i=1}^{n+1} p_i^2 \). The primary constraint is usually written as

\[
\varphi = (q, q) - R^2 = r^2 - R^2 = 0
\]

The Eqs. (2.1) take the form

\[
\{ \varphi, H \} = 2V, \quad \{ V, H \} = 2H
\]

\[
\{ \varphi, V \} = 2(\varphi + R^2) = 2r^2,
\]

where \( V = (q, p) \) is the secondary constraint. The system of differential equations is

\[
-2V \frac{\partial F}{\partial H} - 2(\varphi + R^2) \frac{\partial F}{\partial V} = 0
\]

\[
2V \frac{\partial F}{\partial \varphi} - 2H \frac{\partial F}{\partial V} = 0
\]

\[
2(\varphi + R^2) \frac{\partial F}{\partial \varphi} - 2H \frac{\partial F}{\partial H} = 0.
\]

The condition (2.3) is satisfied the dimension of the matrix being odd. By applying the characteristic method [15, 16] we have from the last equation

\[
\varphi'(t) = \varphi + R^2 \quad H'(t) = -H
\]

The solution is

\[
\varphi + R^2 = e^t \quad H = e^{-t}
\]

By eliminating \( t \) we find that the solution has the form

\[
F = (\varphi + R^2)H + g(V).
\]
If we use this form in the second equation we get \( g(V) = -V^2 / 2 \). Thus the elementary solution of the system (3.1) is

\[
F = UH - V^2 / 2,
\]

where \( U = \phi + R^2 = r^2 = (q, q) \).

The Casimir, i.e. the true Hamiltonian will be

\[
\mathcal{H} = UH - V^2 / 2 = \frac{1}{2} \left( \sum_{i=1}^{n+1} q_i^2 \sum_{j=1}^{n+1} p_j^2 - \sum_{i=1}^{n+1} p_i q_i \right)
\]

\[
= \frac{1}{2} \sum_{i,j} (q_i p_j - q_j p_i)^2 = \frac{1}{2} \sum_{i,j} L_{ij}^2 = \frac{1}{2} I^2
\]

by the Lagrange identity. Thus the quantum observable is the square of the angular momentum a result obtained by many people and also by us by using ad hoc procedure [10]. Similarly to other quantum physical problems one can find a common base of eigenvectors for the old Hamiltonian \( H \) and the new one \( \mathcal{H} \) and as a result one obtains for the eigenvalues of \( H \) the known result \( l(l+n-1)/2R^2, \ l = 1, 2, ..., n = 1, 2, ... \) which has the dimension of energy. This result shows that in the past the physicists have diagonalized the “third” component of the above algebra, result which is sufficient from a pragmatic point of view since in this case the eigenvalues differ only by a numerical constant from those of the Casimir operator[18, 19].

Let us show now that the Casimir \( \mathcal{H} \) is the good classical Hamiltonian of the problem. The Hamilton equations

\[
\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial \mathcal{H}}{\partial q_j}
\]

have the form

\[
\dot{q}_j = p_j (q, q) - q_j (q, p)
\]

\[
\dot{p}_j = -q_j (p, p) + p_j (q, p).
\]

Multiplying the first equation by \( p_j \), the second by \( q_j \) and taking the sum we get

\[
\dot{q}_j p_j + q_j \dot{p}_j = \frac{d}{dt} (q, p) = \frac{dV}{dt} = 0.
\]

Similarly multiplying the first equation by \( q_j \) we obtain

\[
\dot{q}_j q_j = \frac{1}{2} \frac{dU}{dt} = (q, p)(q, q) - (q, q)(q, p) = 0
\]

which shows that \( U \) and \( V \) are constant in time and if the constraints are fulfilled at the initial time they will be fulfilled at any time. We consider the last two re-
lations as a consistency check of the formalism. We remind also that the constraints preservation in time was one of the building blocks of Dirac theory [7].

The above Casimir $\mathcal{H}$ appeared for the first time in our paper [10] although there are a few people who could find it. For example Moser [20] in a different context that of the regularization of the Kepler's $n$-dimensional motion used "half" of our Hamiltonian, i.e. $\frac{1}{2}p^2q^2$ and considered the geodesic flow onto $T(S^n)$, the tangent bundle of the $n$-sphere, defined by the conditions $q^2 = 1$ and $V = (p, q) = 0$.

In a more recent paper, Alber and Marsden [21] studying the semiclassical monodromy for the spherical pendulum used the Hamiltonian

$$H = \frac{1}{2}((p, p) - (p, q)^2) + q_n$$

and they say that the constraint $(q, q) = 1$ has been enforced by the extra term $(p, q)^2$.

In the following we want to show that the classical problem of the hydrogen atom spectrum fits well into this scheme. The classical Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{\kappa}{r},$$

where $m$ is the reduced mass and $\kappa = Ze^2$. In the standard treatment $H$ and the angular momentum $L = r \times p$ are constants of the motion, but these quantities are not enough to make the orbit to be closed, and not enough for having a discrete spectrum for the quantum problem. We quote from Schiff's book [22].

"The rotational symmetry of $H$ is enough to cause the orbit to lie in some plane through $O$, but is not enough to require the orbit to be closed. A small deviation of the potential energy from the newtonian form $V(r) = -\kappa/r$ causes the major axis $PA$ of the ellipse to precess slowly, so that the orbit is not closed. This suggests that there is some quantity, other than $H$ and $L$, that is a constant of the motion and that can be used to characterize the orientation of the major axis in the orbital plane."

Usually one considers that the trajectories are closed because there is a hidden symmetry in the system in question and consequently there are additional integrals of motion. These new integrals that we see as constraints are the components of the Laplace-Runge-Lenz vector. This vector is proportional with the dipole moment of the orbit and has the form

$$M = \frac{p \times L}{m} - \frac{\kappa}{r} r.$$  

On the other hand a direct calculation shows that
\[
M = \frac{4}{3} E |d|, 
\]
where \(d\) is the mean dipole moment of the elliptical motion and \(E\) is the energy. If \(d\) is not bounded the motion is not closed and the possibility of discrete spectrum for the quantum problem disappears. These constraints generate the first quadratic algebra in quantum physics. Indeed we have
\[
\{M_i, M_j\} = -\frac{2}{m} \epsilon_{ijk} HL_k \quad i, j = 1, 2, 3 \tag{3.2}
\]
where \(H\) and \(L\) are the energy and, respectively, the angular momentum. The energy commutes with all the other quantities
\[
\{H, M_i\} = \{H, L_i\} = 0 \quad i = 1, 2, 3 \tag{3.3}
\]
and we have also
\[
\{M_i, L_j\} = \epsilon_{ijk} M_k, \quad \{L_i, L_j\} = \epsilon_{ijk} L_k, \quad i = 1, 2, 3 \tag{3.4}
\]
The above algebra shows that there are seven dynamical variables and that the resulting Lie algebra has a center formed by the generator \(H\), i.e. it is a non semisimple algebra!

The Eqs. (2.1) have the form
\[
-L_3 \frac{\partial F}{\partial L_2} + L_2 \frac{\partial F}{\partial L_3} - M_3 \frac{\partial F}{\partial M_2} + M_2 \frac{\partial F}{\partial M_3} = 0 \\
L_3 \frac{\partial F}{\partial L_1} - L_1 \frac{\partial F}{\partial L_3} + M_3 \frac{\partial F}{\partial M_1} - M_1 \frac{\partial F}{\partial M_3} = 0 \\
-L_2 \frac{\partial F}{\partial L_1} + L_1 \frac{\partial F}{\partial L_2} - M_2 \frac{\partial F}{\partial M_1} + M_1 \frac{\partial F}{\partial M_2} = 0 \tag{3.2}
\]
\[
-M_3 \frac{\partial F}{\partial L_2} + M_2 \frac{\partial F}{\partial L_3} + \alpha L_3 \frac{\partial F}{\partial M_2} - \alpha L_2 \frac{\partial F}{\partial M_3} = 0 \\
M_3 \frac{\partial F}{\partial L_1} - M_1 \frac{\partial F}{\partial L_3} - \alpha L_3 \frac{\partial F}{\partial M_1} + \alpha L_1 \frac{\partial F}{\partial M_3} = 0 \\
-M_2 \frac{\partial F}{\partial L_1} + M_1 \frac{\partial F}{\partial L_2} + \alpha L_2 \frac{\partial F}{\partial M_1} - \alpha L_1 \frac{\partial F}{\partial M_2} = 0,
\]
where \(\alpha = 2H/m\). Since \(H\) commutes with all the other quantities it is in the centre of the algebra and it will be a Casimir, i.e. an observable in the quantum theory. Thus \(C_1 = H = E\) is a good quantum number.

One can easily see that \(L^2\) and \(M^2\) are separately solutions of the first three equations (3.2), but none of them satisfies the last three equations. We look for a solution of the form
\[ F = aL^2 + bM^2 \]

with \( a \) and \( b \) some constants. We find from the fourth equation that \( b/a = -1/\alpha \)
and the second Casimir is

\[ C_2 = a(L^2 - \frac{M^2}{\alpha}) \]

The third one is

\[ C_3 = L \cdot M \]

If we use the quantum form of \( M^2 \), i.e.

\[ M^2 = \frac{2E}{m}(L^2 + \hbar^2) + \kappa^2 \]

and take \( a = 1/4 \) in the second Casimir, we find the known form of the energy levels

\[ E = -\frac{m\kappa^2}{2\hbar^2(2\kappa + 1)^2} \]

where \( \kappa = 0, 1/2, 1, ... \) and the eigenvalues of the quadratic operator \( C_2 \) are given as usual by \( c(c+1) \).

The commutation of \( H \) with all the other generators of the algebra gives us the possibility of replacing \( H \) by \( E \) for an eigenvalue corresponding to a bound state as it is done in the usual treatment of the problem since then \( E \) has a definite constant value. In conclusion the hydrogen atom has a symmetry which is given in terms of a non semisimple Lie algebra of dimension seven. Of course the above algebra can be viewed as a central extension of the \( O(4) \) symmetry, so the known things do not change, the spectrum is the same as it should be but we consider that its deduction is simpler in our approach. Of course, although the angular momentum is not between the Casimirs of the algebra, we can find a common set of eigenfunctions for \( H \) and \( L^2 \) since \( \{ H, L \} = 0 \) and all the usual results remain unchanged. Also we can find a common eigenfunctions set for \( H \) and \( M^2 \).

One more remark concerns the geodesic flow on the sphere \( S^n \) regarded as a submanifold of the \( (n+1) \)-dimensional space and the \( n \)-dimensional Kepler problem. It was shown by Moser [20] that by using a stereographic projection of the punctured sphere \( S^n \) onto an \( n \)-dimensional Euclidean space one obtains the Kepler Hamiltonian from the Hamiltonian \( \mathcal{H} \). In other words the above two problems are the same and solving one of them we obtain the solution for the other.

The non semisimple algebras also appeared recently in other areas of physics such as string backgrounds or WZW models [23, 13].

In the same manner can be studied the quantisation of the \( n \)-dimensional isotropic oscillator when \( n > 1 \). In this case the constraints assuring the closing of the classical three-dimensional orbit are expressed in terms of a quadrupole tensor [24, 22].
4. OTHER RESULTS

The investigation of the Poisson-Lie structures is only at its beginning [11] and there are sound reasons for such an interest. It has been shown that the classical limit of a quantum group is a Poisson-Lie group, or in other words a Poisson-Lie structure [25], opening the way for applications to integrable systems. In the following we consider a little more complicated structure, the functions $f_{ij}$ entering Eqs. (2.1) being quadratic functions and show how our equations (2.1)--(2.3) are useful in finding the Casimirs of the corresponding algebras, i.e. the conserved quantities of the physical models described by the corresponding Lie algebras.

One of the best known such a structure is that introduced by Sklyanin in connection with the Yang-Baxter equations [12]. The Eqs. (2.1) have the following form

$$
\begin{align*}
\{u_2, u_1\} &= b_1 u_3 u_4 \\
\{u_3, u_1\} &= b_2 u_2 u_4 \\
\{u_4, u_1\} &= b_3 u_2 u_3 \\
\{u_5, u_1\} &= a_1 u_1 u_4 \\
\{u_6, u_1\} &= a_2 u_1 u_3 \\
\{u_7, u_1\} &= a_3 u_1 u_2
\end{align*}
$$

$a_i$ and $b_i$, $i = 1, 2, 3$ being arbitrary complex numbers. The case considered by Sklyanin was $a_1 = -a_2 = a_3$ and $b_1 + b_2 + b_3 = 0$.

The condition (2.3) is equivalent to

$$a_1 b_1 - a_2 b_2 + a_3 b_3 = 0 \quad (3.3)$$

so in the following we suppose that (3.3) holds. From the equations (2.2) we find that the following quantities are Casimirs

$$
\begin{align*}
C_1 &= a_1 u_1^2 - b_2 u_2^3 + b_1 u_3^2 \\
C_2 &= a_2 u_1^2 - b_3 u_2^3 + b_2 u_3^2
\end{align*}
$$

Another quadratic algebra is found in ref. [26] used to describe the kinematical symmetry of a spin chain on a one dimensional lattice. It has the form

$$
\begin{align*}
\{u_1, u_2\} &= -\frac{a_1}{2} u_2^2 \\
\{u_1, u_3\} &= a u_1 \\
\{u_2, u_3\} &= a u_2 \\
\{u_4, u_i\} &= 0 \quad i = 1, 2, 3
\end{align*}
$$

(3.4)

Since $u_4$ commutes with the other generators a solution of the eqs. (2.2)) is of the form $f(u_4)$ with $f$ an arbitrary derivable function. The other Casimir is

$$C = u_1 u_2^{-1} - \frac{1}{2} u_3^2.$$

If we perturb the second equation (3.4) to the following form
\{u_1, u_3\} = au_1 + bu_2

obtaining a Poisson-Lie structure on the 2-dimensional Galilei algebra [27], the Casimir is more complicated and cannot be guessed simply. The characteristic method gives

\[ C = au_1 u_2^{-1} - b \log |u_2| - \frac{a}{2} u_3. \]

Another interesting example appears in the construction of Wess-Zumino-Witten models on non semisimple groups [13], see also [23]. The algebra has the following structure

\[ \{J, P_i\} = \epsilon_{ij} P_j, \quad \{P_i, P_j\} = \epsilon_{ij} T, \]

\[ \{T, J\} = \{T, P_i\} = 0, \quad i = 1, 2 \]

In general, given a Lie algebra to define a WZW model one needs a bilinear form in the generators of the algebra, form which is symmetric, invariant and nondegenerate. Usually for semisimple groups one takes \( \text{Tr} u_i u_i \), with the trace taken in the adjoint representation of the group. For non semisimple groups this quadratic form is degenerate. By applying our formalism one finds easily the two Casimirs

\[ C_1 = P_1^2 + P_2^2 + 2JT, \quad C_2 = g(T), \]

where \( g(T) \) is an arbitrary derivable function of \( T \). Thus the most general bilinear form is

\[ \Omega = a(P_1^2 + P_2^2 + 2JT) + bT^2, \]

where \( a \) and \( b \) are two arbitrary constants, which is the result of Nappi and Witten[13].

5. CONCLUSION

The Dirac quantum theory [7] was patterned after the classical theory, the “observables” representing constraints must have zero expectation values. This requirement is not consistent with the fact that the Poisson brackets between Hamiltonian and constraints and between constraints themselves may not vanish such as Eqs. (2.1) show. In this paper we have shown that this inconsistency disappears if we view the Hamiltonian and the constraints as generators of a larger symmetry. The observables will be the Casimir(s) of this new Poisson-Lie algebra. In this approach the Hamiltonian has its own rôle but much diminished than in the case of no constraints. Of course, one can find a common basis of eigenvectors for the Hamiltonian and the Casimir(s) and thus its predicting power is still considerable. However there could be cases when starting with a Dirac form Hamiltonian
\[ H_D = H + \mu_i \phi_i \]

may be misleading and cause troubles when using it for the description of physical systems, the true Hamiltonians being more complicated functions of both the old Hamiltonian and the constraints together, as the above examples suggest. The lesson to be learnt is that for constrained systems almost no one of the initial dynamical variables transforms into an observable. In this respect the hydrogen atom is an exception, the reason being that the classical Hamiltonian commutes with all the constraints, being in the centre of the Poisson-Lie algebra.

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