NEW ASPECTS IN INTERACTION OF GRAVITATIONAL WAVES

BRÎNDUȘA CIOBANU*

Physics Department, Faculty of Machine Manufacturing,
“Gh. Asachi” Technical University, Iași, Romania

Received February 21, 2005

In this paper we assume the following model: the existence of each gravitational field is equivalent to the existence of some space-time manifold $V_4$. In this context, even under the hypothesis of some linear approximation, a gravitational wave corresponds to a non-Euclidean type structure of $V_4$. In order to study the interaction between two gravitational waves, we consider a second wave, also described in the linear approximation, wave to be “deformed” by the non-Euclidean structure of $V_4$. This “deformation” will characterize the interaction between the two waves.

It is well known that in weak gravitational fields, in a first approximation, Einstein’s equations become Maxwell type equations. Under this hypothesis, we study the interaction of two gravitational waves by rewriting the gravitational Maxwell type equations under some induced metric by one of the assumed waves. Respectively, we determine the formulas of the gravitational permittivity and gravitational permeability. In the special case of a gravitational plane wave, we’ll also deduce and interpret the wave equation.

1. INTRODUCTION

The existence of the gravitational waves is an important test for the General Relativity Theory. Nevertheless all the efforts for detecting the effects induced by the gravitational radiation as a result of its interaction with the matter are less conclusive by now, mainly because of gravitational radiation intensity low value.

All the results already published in different papers open a new way of approaching the gravitational waves interaction. The present paper contains the following steps:

– the Einstein equations are writing in the linear approximation under the form of gravitational field Maxwell type equations;
– the transforming of the gravitational field Maxwell type equations in the corresponding metric of a monochromatic gravitational plane wave. Consequently, the new equations will describe the interaction between the two gravitational waves;

* bciobanu2003@yahoo.com

– the gravitational permittivity and gravitational permeability deduction;
– the wave equations’ deduction and interpretation of the gravitoelectrical component in the assumed metric.

The gravitational field Maxwell type equations set describing a gravitational wave corresponds to a non-Euclidean structure for $V_4$. The action of these equations on another gravitational wave metric background leads to an initial space-time structure “distortion”. Hence we can speak about an interaction between the two waves through the non-Euclidean character of $V_4$. This will be directly reflected in the non-linearity of the equation for the gravitational wave propagation.

We must mention here that in the linear theories the waves do not interact. Yet in our case we may speak about two gravitational waves interaction (initially regarded as being linear) through the non-Euclidean character of the $V_4$ quadrispace.

In literature, this way of approaching the subject is not singular. Therefore we can meet it in reference [1], for the gravitomagnetic generator research in arbitrary coordinates, being described as “a possible way to study the gravitational waves interaction”.

Consider a monochromatic gravitational plane wave characterized by the following metric:

$$ds^2 = dx^2 + [1 + g \sin(kx - \omega t)]dy^2 + [1 - g \sin(kx - \omega t)]dz^2 - c^2dt^2$$  \hspace{1cm} (1)

g being wave’s amplitude, and $\omega$ its pulsation. We mention that the assumed metric can be found in some recent papers [2], [3], discussing the gravitational waves interaction with the cosmic plasma and electromagnetic field respectively.

The covariant and, respectively, contravariant non-vanishing components of the metric tensor associated to the gravitational wave are:

\begin{align}
g_{00} &= -1 \\
g_{11} &= 1 \\
g_{22} &= 1 + g \sin(kx - \omega t) \\
g_{33} &= 1 - g \sin(kx - \omega t) \\
g^{00} &= -1 \\
g^{11} &= 1 \\
g^{22} &= [1 + g \sin(kx - \omega t)]^{-1} \\
g^{33} &= [1 - g \sin(kx - \omega t)]^{-1}
\end{align}  \hspace{1cm} (2)

The non-vanishing Christoffel symbols are given by the following formulae:
2. THE GRAVITATIONAL PERMITTIVITY AND GRAVITATIONAL PERMEABILITY TENSORS

According to references [3–6], the gravitoelectric induction “vector” in the presence of the gravitational field generated by the gravitational wave has the following expression:

\[
\bar{D}^\mu = (-\det g_{ab})^{1/2}(\bar{u}_i g^{a1} + \bar{u}_2 g^{a2} + \bar{u}_3 g^{a3})g^{b0}F_{ab}^\mu
\]  

(5)

where \(\bar{u}_i\) (\(i = 1, 2, 3\)) are the versors of the three-dimensional space coordinate system and \(F_{ab}^\mu\) represents the components of the gravitoelectromagnetic field tensor.

According to [6] and [7] on the generalization of the electrodynamics theory, as formulated by Wilson-Dicke [8], we’ll determine the components of the gravitational permittivity tensor and those of the gravitational permeability tensor respectively.

Thus, having in mind the formula of the gravitoelectric field intensity vector

\[
\bar{g}^\mu = F_{01}^\mu \bar{u}_1 + F_{02}^\mu \bar{u}_2 + F_{03}^\mu \bar{u}_3,
\]

(6)

and the link relations between \(\bar{g}^\mu\) and \(\bar{D}^\mu\)

\[
\begin{bmatrix}
D_1^\mu \\
D_2^\mu \\
D_3^\mu
\end{bmatrix} = \\
\begin{bmatrix}
1 - g^2 \sin^2(kx - \omega t) \right)^{1/2} & 0 & 0 \\
0 & \left[1 - g \sin(kx - \omega t) \right]^{1/2} & 0 \\
0 & 0 & \left[1 + g \sin(kx - \omega t) \right]^{1/2}
\end{bmatrix}
\]

(7)

\[
\begin{bmatrix}
g_1^\mu \\
g_2^\mu \\
g_3^\mu
\end{bmatrix}
\]
through identification, the following components of the permittivity tensor of the gravitational space with monochromatic gravitational plane wave result:

\[
\begin{align*}
\varepsilon_{11} &= [1 - g^2 \sin^2 (kx - \omega t)]^{1/2} \\
\varepsilon_{22} &= \left[\frac{1 - g \sin(kx - \omega t)}{1 + g \sin(kx - \omega t)}\right]^{1/2} \\
\varepsilon_{33} &= \left[\frac{1 + g \sin(kx - \omega t)}{1 - g \sin(kx - \omega t)}\right]^{1/2}
\end{align*}
\] (8)

The gravitational permeability tensor’s components can be obtained through a similar reasoning. Thus, according to [6], [7], the intensity vector of the gravitomagnetic field can be written as follows:

\[
\tilde{H}_g^\mu = [1 - g^2 \sin^2 (kx - \omega t)]^{1/2} (\tilde{u}_1 g_{33}^\mu F_{32}^\mu + \tilde{u}_2 g_{11}^\mu F_{13}^\mu + \tilde{u}_3 g_{22}^\mu F_{21}^\mu)
\] (10)

or

\[
\tilde{H}_g^\mu = [1 - g^2 \sin^2 (kx - \omega t)]^{-1/2} B_{g1}^\mu \tilde{u}_1 + \left[\frac{1 + g \sin(kx - \omega t)}{1 - g \sin(kx - \omega t)}\right]^{1/2} B_{g2}^\mu \tilde{u}_2 + \left[\frac{1 - g \sin(kx - \omega t)}{1 + g \sin(kx - \omega t)}\right]^{1/2} B_{g3}^\mu \tilde{u}_3
\] (11)

Then the following expressions are obtained:

\[
\begin{align*}
B_{g1}^\mu &= [1 - g^2 \sin^2 (kx - \omega t)]^{-1/2} H_{g1}^\mu \\
B_{g2}^\mu &= \left[\frac{1 - g \sin(kx - \omega t)}{1 + g \sin(kx - \omega t)}\right]^{1/2} H_{g2}^\mu \\
B_{g3}^\mu &= \left[\frac{1 + g \sin(kx - \omega t)}{1 - g \sin(kx - \omega t)}\right]^{1/2} H_{g3}^\mu
\end{align*}
\] (12)

written in the matrix form

\[
\begin{pmatrix}
B_{g1}^\mu \\
B_{g2}^\mu \\
B_{g3}^\mu
\end{pmatrix} =
\begin{pmatrix}
\mu_{11} & 0 & 0 \\
0 & \mu_{22} & 0 \\
0 & 0 & \mu_{33}
\end{pmatrix}
\begin{pmatrix}
H_{g1}^\mu \\
H_{g2}^\mu \\
H_{g3}^\mu
\end{pmatrix}
\] (13)
the diagonal components of the gravitational permeability tensor are given by:

\[
\begin{align*}
\mu_{11} &= [1 - g^2 \sin^2(kt - \omega t)]^{1/2} \\
\mu_{22} &= \left[\frac{1 - g \sin(kx - \omega t)}{1 + g \sin(kx - \omega t)}\right]^{1/2} \\
\mu_{33} &= \left[\frac{1 + g \sin(kx - \omega t)}{1 - g \sin(kx - \omega t)}\right]^{1/2}
\end{align*}
\]

(14)

Hence, our study underlays the following preliminary conclusions:

– on the already considered metric, under the presence of the mass associated to the gravitational wave, the vacuum behaves like a dielectric anisotropic medium characterized by the gravitational permittivity and the gravitational permeability, given in (9) and (14);

– the anisotropy of the gravitational field expressed in the metric’s expression (1) induces a vacuum’s anisotropy;

– Dicke’s property, according to which \( \varepsilon_{ij} = \mu_{ij} \), is already fulfilled.

### 3. WAVE EQUATIONS

Next we shall deduce the wave equation for the \( G^\mu \) component in the metric (1).

With this purpose in view, we consider the “sourceless” \( (T^{\mu0} = 0, \ \vec{T}_\mu = 0) \) Maxwell type gravitational equations [9, 10]:

\[
\nabla \times \vec{g}^\mu = -\frac{\partial \vec{B}_g^g}{\partial t}
\]

(15)

\[
\nabla \cdot \vec{B}_g^g = 0
\]

(16)

\[
\nabla \cdot \vec{D}^g = 0
\]

(17)

\[
\nabla \times \vec{H}_g^\mu = \frac{1}{c} \frac{\partial \vec{D}_g^g}{\partial t}
\]

(18)

By applying the curl operator to (15) we get:

\[
\Delta \vec{g}^\mu = \ddot{\mu}_j \left[ \partial_j \partial_i g^\mu_{ij} + \varepsilon_{jk} \partial_k \partial_i (\mu_{lm} H^\mu_{gm}) \right]
\]

(19)

where \( \partial_k = \partial / \partial x^k \) and \( \partial_i = \partial / \partial t_i \).

Multiplying by \( \mu_{ij}^{-1} \) and re-noting the indices, the equation (17) becomes:
\[ \partial_j g_i^\mu = -\tilde{\mu}_j g_p^\mu \partial_r \mu_{rp} \]  

(20)

From (19) and (20), by applying the metric (1), we get:

\[ \Delta g^\mu = \tilde{u}_j \left[ \tilde{\partial}_i \left( \tilde{\mu}_j g_p^\mu \tilde{\partial}_r \mu_{rp} \right) \right] + \tilde{u}_j \left[ e_{j21} \tilde{\partial}_i \left( \mu_{11} \tilde{\partial}_2 H_{g1}^\mu \right) + e_{j31} \tilde{\partial}_i \left( \mu_{11} \tilde{\partial}_3 H_{g1}^\mu \right) + 
  + e_{j12} \tilde{\partial}_i \left( \mu_{22} \tilde{\partial}_2 H_{g2}^\mu \right) + e_{j22} \tilde{\partial}_i \left( \mu_{22} \tilde{\partial}_3 H_{g2}^\mu \right) + e_{j32} \tilde{\partial}_i \left( \mu_{33} \tilde{\partial}_3 H_{g2}^\mu \right) \right] 
  + e_{j13} \tilde{\partial}_i \left( \mu_{33} \tilde{\partial}_2 H_{g3}^\mu \right) + e_{j23} \tilde{\partial}_i \left( \mu_{33} \tilde{\partial}_3 H_{g3}^\mu \right) \]

(21)

From (18) equation, multiplying by Levi-Civita symbol, with the hypothesis of metric (1), one obtains:

\[ \tilde{\partial}_m H_{gm}^\mu - \tilde{\partial}_m H_{gm}^\mu = \frac{1}{c} \left[ e_{1mn} \tilde{\partial}_i \left( \mu_{11} g_i^m \right) + e_{2mn} \tilde{\partial}_i \left( \mu_{22} g_i^m \right) + e_{3mn} \tilde{\partial}_i \left( \mu_{33} g_i^m \right) \right] \]

(22)

Through successive transformations equation (16) leads to:

\[ H_{g1}^\mu \tilde{\partial}_i \mu_{11} = -(\mu_{11} \tilde{\partial}_1 H_{g1}^\mu + \mu_{22} \tilde{\partial}_2 H_{g1}^\mu + \mu_{33} \tilde{\partial}_3 H_{g1}^\mu) \]

(23)

Let us consider the particular case:

\[ H_{g2}^\mu = 0, \quad H_{g3}^\mu = 0 \]

(24)

In this context, from (21) we find:

\[ \Delta g^\mu = \tilde{u}_j \left[ \tilde{\partial}_i \left( -\tilde{\mu}_j g_p^\mu \tilde{\partial}_r \mu_{rp} \right) \right] + \tilde{u}_j \frac{1}{c} \left[ e_{j21} \tilde{\partial}_i \left( \mu_{11} \tilde{\partial}_2 H_{g1}^\mu \right) + e_{j31} \tilde{\partial}_i \left( \mu_{11} \tilde{\partial}_3 H_{g1}^\mu \right) \right] \]

(25)

and (22) leads to:

\[ \partial_2 H_{g1} = -\frac{1}{c} \tilde{\partial}_i \left( \mu_{33} g_i^2 \right) \]

(26)

Substituting (26) in (25) and through some successive transformations one gets:

\[ \Delta g^\mu = \tilde{u}_j \left[ -g_p^\mu \tilde{\partial}_1 \left( \frac{\mu_{11}}{\mu_{11}} \right) - \tilde{\partial}_1 \left( \frac{\mu_{11}}{\mu_{11}} \right) \right] + 
  + \frac{1}{c^2} \left[ \mu_{11} \tilde{\partial}_2 \tilde{g}_2^\mu + \left( \mu_{11} \mu_{22} \tilde{g}_2^\mu + (\mu_{11} \mu_{22} + 2 \mu_{11} \mu_{22}) \tilde{g}_2^\mu \right) \right] + 
  + \left( \mu_{11} \mu_{22} + 2 \mu_{11} \mu_{22} \right) \tilde{g}_3^\mu \]

(27)
We’ll study the special situation when
\[ g_1^\mu = 0, \quad g_2^\mu = 0 \]  

(28)

By keeping only the first order terms in \( g \) \((g \ll 1)\), (27) becomes:
\[
\Delta g_3^\mu - \frac{\mu_1 \mu_3}{c^2} \frac{\partial^2 g_3^\mu}{\partial t^2} = - \frac{2(1 \mu \cos(kx - \omega t)}{c^2[1 - g \sin(kx - \omega t)]} \cdot \frac{\partial g_3^\mu}{\partial t} - \frac{1 \mu \omega^2 g \sin(kx - \omega t)}{c^2[1 - g \sin(kx - \omega t)]} g_3^0 \]  

(29)

Analyzing the equation (28), the following preliminary conclusions can be drawn out:

– because of the right hand-side terms in the relation (28), it does not describe a free gravitational wave. By comparing it to the wave equation for the gravitoelectric component (for some homogene perturbations, \( \forall T^{\mu0} = 0 \)), \([11]\), the first term corresponds to some interaction between the gravitational drift current, induced by the gravitational wave, \( \partial g_3^\mu / \partial t \), and the “medium” corresponding to the supposed metric. The latter interacts through the gravitational conductivity \([12]\)

\[
\sigma_g = - \frac{2(1 \mu \cos(kx - \omega t)}{c^2[1 - g \sin(kx - \omega t)]} \]  

(30)

The second term in the right hand-side of (28), written in the form \([13], [14]\):

\[
\frac{1}{2} \omega \rho g(kx - \omega t)T^\mu \]  

(31)

expresses an interaction between the “induction gravitational current”, induced by the \( T^\mu \) gravitational wave, and the “medium”, through \( \omega \) pulsation and \( \rho g(kx - \omega t) \);

– identifying the coefficients, between (28) and the general wave equation, the gravitational wave’s speed for metric (1) is given by:

\[
\nu = \frac{c}{\sqrt{\mu_1 \mu_2}} \]  

(32)

It can be observed that formula (32) differs from light’s speed expression in vacuum. Thus the gravitational field induced by the metric behaves like a medium with the refraction index

\[
n = \sqrt{\mu_1 \mu_2} \]  

(32)

The refraction index depends both on the \( x \) spatial coordinate, as well as on the temporal coordinate \( t \).
4. CONCLUSIONS

In a first approximation, Einstein’s equations become Maxwell type equations. Under this hypothesis, the interaction of two gravitational waves was studied, namely:

- on the induced metric by one of the assumed waves, under the presence of the mass associated to the gravitational wave, the vacuum behaves like a dielectric anisotropic medium, characterized by a gravitational permittivity and a gravitational permeability;
- the anisotropy of gravitational field expressed in the metric induces a vacuum’s anisotropy;
- Dicke’s property for gravitational permittivity components and the gravitational permeability components is fulfilled.

We have shown that the wave equation does not describe a free gravitational wave. By comparing it to the wave’s equation for the gravitoelectric component, the first term corresponds to some interaction between the gravitational drift current and the “medium” corresponding to the supposed metric.

In this way the speed of the gravitational wave and the gravitational refraction index were determined.

REFERENCES