The paper analyses the cases of integrability for bi-dimensional systems with polynomial autonomous Hamiltonians. We use an algorithm which allows a direct computation of a second invariant for the system, starting from an imposed form for it, proportional with various powers of velocities. Two concrete examples, the mechanical Yang-Mills and the Hénon-Heiles systems, will be effectively studied and the integrability cases will be recovered.

1. INTRODUCTION

The simplest meaning of the “integrability” of dynamical systems consists in the existence of some analytical and time-independent quantities, which are in involution and whose number is equal with the number of degrees of freedom of the system. For an integrable system with \( n \) degrees of freedom, described in the phase space \( \{x^1, ..., x^n, p_1, ..., p_n\} \) by the Hamiltonian \( H(x^1, ..., x^n, p_1, ..., p_n) \), apart from the Hamiltonian, there are other \( n - 1 \) constants \( \{C_i; i = 1, ..., n - 1\} \) so that:

\[
\frac{dC_i}{dt} = \{H, C_i\} = 0; \quad \{C_i, C_j\} = 0 \quad (1)
\]

There are also indirect approaches of integrability: a) the Painlevé analysis who consists of requiring that the general solutions should have the Painlevé property, \( i.e., \) their only movable singularities on the complex time plane are poles [1]; b) the Lax-pair method who searches for two operators, \( L \) and \( M \), so that \( \frac{dL}{dt} = [L, M] \) should be equivalent to the original Hamiltonian’s equations of motions. If this could be done then the coefficients of \( l^n \) in the expansion of \( \det(L - II) \) would be invariants and in involution [2].

The way chosen in this paper is the direct computation of the invariants [3]. When we are looking for exact solutions of the equations of motion for Hamiltonian systems, the approach could depend on:
a) the type of Hamiltonian: (kinetic energy + potential energy/ terms linear in velocity);

b) the type of potential: (homogeneous in variables/ non-homogeneous/ non-polynomial);

c) the type of invariants: (linear in velocity/ quadratic/ etc.).

We restrict ourself to the study of the 2-dimensional autonomous Hamiltonian systems. In this case, the Hamiltonian itself is a first constant of the motion. The only problem to be solved in order to fulfill the full integrability condition is to find a second invariant.

The paper consists in the following sections: section 2 will deal with the general algorithm for finding the second invariant of the system. We will investigate both ways: (i) by imposing a specific form of the invariant we will determine the potentials which accept such type of invariant; (ii) starting from a concrete potential we will find the invariant. A concrete example will be presented in the third section of the paper. It will consist in the analysis of two dynamical systems with polynomial potentials: the Yang-Mills and Hénon-Heiles systems. Some concluding remarks will end the paper.

2. GENERAL ANALYSIS OF INTEGRABILITY

Let us consider a dynamical Hamiltonian system of the form:

\[ H(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x, y) \]  (2)

where the potential \( V(x, y) \) has a polynomial expression.

The equations of motion for the system (2) are:

\[
\begin{align*}
\dot{x} &= -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} \\
\dot{y} &= -\frac{\partial H}{\partial y} = -\frac{\partial V}{\partial y}
\end{align*}
\]  (3)

In order to give a hint on how to find a second invariant of the system (2), we are looking for the invariant of the form:

\[ C = f_1 \dot{x}^2 + f_2 \dot{y}^2 + g_1 \dot{x} \dot{y} + g_2 \]  (4)

where \( f_i = f_i(x, y), \ g_i = g_i(x, y), \ i = 1, 2, \) should be determined.

Remark: \( C \) may not contain linear terms of velocities because of the time reflection symmetry.

The integral of motion \( C \) exist if the following condition is respected:
In the relation (5), by imposing to the coefficients of the terms containing third order of velocities to be equal to zero, we obtain the differential system:

\[
\frac{\partial f_1}{\partial x} = 0; \quad \frac{\partial f_2}{\partial y} = 0; \quad \frac{\partial f_1}{\partial y} + \frac{\partial g_1}{\partial x} = 0; \quad \frac{\partial f_2}{\partial x} + \frac{\partial g_1}{\partial y} = 0
\] (6)

The solution of the system (6) is:

\[
\begin{align*}
 f_1 &= my^2 + ky + \nu \\
 f_2 &= mx^2 + px + s \\
 g_1(x, y) &= -2mxy - kx - py + u
\end{align*}
\] (7)

where \( m, k, p, \nu, s, p, u \) are arbitrary constants.

Also, through cancelling the coefficients of linear terms of velocities in the condition (5), we obtain the following system:

\[
\begin{align*}
 2f_1\ddot{x} + g_1\ddot{y} + \frac{\partial g_2}{\partial x} &= 0 \\
 2f_2\ddot{y} + g_1\ddot{x} + \frac{\partial g_2}{\partial y} &= 0
\end{align*}
\] (8)

By the substitution of the expressions (7) of \( f_i, \ i = 1, 2, g_1 \) and of the equations of motion (3) into the system (8), we can give the explicit form of \( g_2 \).

The compatibility condition for (8) is of the form:

\[
-\frac{\partial^2 g_2}{\partial x \partial y} = \frac{\partial}{\partial y}[2f_1\ddot{x} + g_1\ddot{y}] = \frac{\partial}{\partial x}[2f_2\ddot{y} + g_1\ddot{x}]
\] (9)

For a Hamiltonian system with the potential function \( V(x, y) \), if we should introduce the relations (7) and the Hamilton equations (3) into the condition (9) and by cancelling the coefficients of the terms which appear, we would obtain a set of relations between the arbitrary constants \( m, k, p, \nu, s, p, u \) and also relations between the constants from the general form of potential function \( V(x, y) \).

As a conclusion, we can find the integrability cases for a given general bi-dimensional dynamical system and the second invariant with quadratic velocities in each of the integrable cases. The same algorithm could be followed when cubic or quartic invariants in velocities are investigated. For example, in the case of a quartic invariant of the form:
\[ C = f_0 \dot{x}^4 + f_1 \dot{x}^3 \dot{y} + f_2 \dot{x}^2 \dot{y}^2 + f_3 \dot{x} \dot{y}^3 + f_4 \dot{y}^4 + g_0 \dot{x}^2 + g_1 \dot{x} \dot{y} + g_2 \dot{y}^2 + h \quad (10) \]

the previous algorithm gives:

\[ f_0(y) = \alpha y^4 + \beta y^3 + \gamma y^2 + \delta y + \varepsilon \]

\[ f_1(x, y) = -(4\alpha y^3 + 3\beta y^2 + 2\gamma y - \delta)x - \xi y^3 + \phi y^2 + \theta y + \rho \]

\[ f_2(x, y) = (6\alpha y^2 + 3\beta y + \gamma)x^2 - (-3\xi y^2 + 2\phi y + \Theta)x + \chi y^2 + \kappa y + \sigma \quad (11) \]

\[ f_3(x, y) = -(4\alpha y + \beta)x^3 + (-6\xi y + 2\phi)x^2 - (2\chi y + \kappa)x - \eta y + \mu \]

\[ f_4(x) = \alpha x^4 + \xi x^3 + \chi x^2 + \eta x + \lambda \]

By imposing the invariance condition for \( C \) and by cancelling the coefficients of the terms with cubic velocities we obtain the system:

\[
\begin{align*}
4f_0\ddot{x} + f_1\ddot{y} + \frac{\partial g_0}{\partial x} &= 0 \\
3f_1\ddot{x} + 2f_2\ddot{y} + \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} &= 0 \\
2f_2\ddot{x} + 3f_3\ddot{y} + \frac{\partial g_1}{\partial y} + \frac{\partial g_2}{\partial x} &= 0 \\
f_3\ddot{x} + 4f_4\ddot{y} + \frac{\partial g_2}{\partial y} &= 0 
\end{align*}
\quad (12)
\]

The equations (12) are integrable if the potential \( V(x, y) \) satisfies to the compatibility condition:

\[
0 = -\frac{\partial^3}{\partial x^3} (3f_1\ddot{x} + 2f_2\ddot{y}) + \frac{\partial^3}{\partial x^3 \partial y} (2f_2\ddot{x} + 3f_3\ddot{y}) - \frac{\partial^3}{\partial x^3 \partial y^2} (3f_1\ddot{x} + 2f_2\ddot{y}) + \frac{\partial^3}{\partial y^3} (4f_0\ddot{x} + f_1\ddot{y}) 
\quad (13)
\]

In addition, when we ask that the coefficients of the linear terms of velocities generated by \( \{H, C\} = 0 \) should be cancelled, we would obtain:

\[
\begin{align*}
2g_0\ddot{x} + g_1\ddot{y} + \frac{\partial h}{\partial x} &= 0 \\
g_1\ddot{x} + 2g_2\ddot{y} + \frac{\partial h}{\partial y} &= 0 
\end{align*}
\quad (14)
\]

The compatibility condition is:

\[
-\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial y} (2g_0\ddot{x} + g_1\ddot{y}) = \frac{\partial}{\partial x} (g_1\ddot{x} + 2g_2\ddot{y}) 
\quad (15)
\]
where \( \dot{x} = -\frac{\partial V(x, y)}{\partial x}, \dot{y} = -\frac{\partial V(x, y)}{\partial y} \).

In conclusion, the functions \( f_i(x, y), i = 0, 4, g_j(x, y), j = 0, 2 \) and \( h(x, y) \) can be determined and the form with quartic velocities of the invariant \( C \), too.

3. EXAMPLES

3.1. NON-HOMOGENEOUS POTENTIALS: THE MECHANICAL YANG-MILLS MODEL

For the bi-dimensional Yang-Mills model, the Hamiltonian function has the general expression:

\[
H = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 - \frac{A}{2} x^2 - \frac{B}{2} y^2 + ax^2 y^2 + bx^4 + y^4
\]

with \( A, B, a, b \) arbitrary constants.

The equations of motion have the form:

\[
\begin{align*}
\ddot{x} &= -\frac{\partial V(x, y)}{\partial x} = Ax - 2ay^2 x - 4bx^3 \\
\ddot{y} &= -\frac{\partial V(x, y)}{\partial y} = By - 2ayx^2 - 4y^3
\end{align*}
\]

Using the equations (17), we have:

\[
(A - B)xy + 2(2 - a)xy^3 + 2(a - 2b)yx^3 = 0
\]

The relation (18) is satisfied iff:

\[
\begin{align*}
A &= B \\
a &= 2 \\
a &= 2b
\end{align*}
\]

It is simple to verify that the system (16) admits a constant of motion linear in velocities which is the angular momentum:

\[
C = y\dot{x} - x\dot{y}
\]

and, in this case, the associated Hamiltonian function is:

\[
H = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 - \frac{A}{2} x^2 - \frac{A}{2} y^2 + 2x^2 y^2 + x^4 + y^4
\]

We will search for the cases of integrability for the Yang-Mills system (16) in which the integrals of motion have quadratic velocities and also the concrete forms of the two invariants \( C \) and \( H \) for each case.
Thus, the study gives the following set of integrable cases for Yang-Mills system:

i) \[ H_1 = \frac{1}{2} x^2 + \frac{1}{2} y^2 - \frac{A}{2} x^2 - \frac{B}{2} y^2 + ax^2 y^2 + bx^4 + y^4 \]
\[ C_1 = 2vH - H \Rightarrow \text{trivial case}; \]  

\[ (22) \]

ii) \[ H_2 = \frac{1}{2} x^2 + \frac{1}{2} y^2 - \frac{A}{2} x^2 - 2Ay^2 + \frac{3}{4} x^2 y^2 + \frac{3}{48} x^4 + y^4, \]
\[ C_2 = k \left( yx^2 - xx^2y + A x^2 y - \frac{1}{2} x^2 y^3 - \frac{1}{4} x^4 y \right) + 2vH; \]
\[ v = ct \]  

\[ (23) \]

iii) \[ H_3 = \frac{1}{2} x^2 + \frac{1}{2} y^2 - \frac{A}{2} x^2 - \frac{A}{8} y^2 + 12x^2 y^2 + 16x^4 + y^4, \]
\[ C_3 = 4p \left( \frac{1}{4} x y^2 - \frac{1}{4} x x^2y + \frac{A}{16} xy^2 - 2x^3 y^2 - xy^4 \right) + 2vH \]  

\[ (24) \]

iv) \[ H_4 = \frac{1}{2} x^2 + \frac{1}{2} y^2 - \frac{A}{2} x^2 - \frac{A}{2} y^2 + 6x^2 y^2 + x^4 + y^4, \]
\[ C_4 = u(x y - A x y + 4x^3 y + 4x y^3) + H; \]  

\[ (25) \]

All these cases were obtained in [4], [5] by other algorithms.

3.2. NON-HOMOGENEOUS POTENTIALS: THE HÉNON-HEILES SYSTEM

We will study the general Hénon-Heiles model described by the following Hamiltonian function:
\[ H = \frac{1}{2} x^2 + \frac{1}{2} y^2 + \frac{A}{2} x^2 + \frac{B}{2} y^2 + axy^2 - bx^3 \]  

\[ (26) \]

with \( A, B, a, b \) arbitrary constants.

The evolution equations have the form:
\[ \ddot{x} = -\frac{\partial H(x, y)}{\partial x} = -Ax - ay^2 + bx^2 \]
\[ \ddot{y} = -\frac{\partial H(x, y)}{\partial y} = -By - 2axy \]  

\[ (27) \]

We start by studying the integrability cases for the Hénon-Heiles model which admits a second invariant \( C = f_1(x^2 + f_2 y^2 + g_1 x y + g_2 \) with quadratic velocities for each integrable case of the dynamical system (26). Using the relations (7) of \( f_i(x, y), i = 1, 2 \), the form (8) of \( g_1(x, y) \), the concrete computation gives two nontrivial cases of integrability:
The explicit forms of the second invariants (4) associated to the integrable Hamiltonians (28), (29) are:

\[ i) \quad C_1 = u x y + A x y + 2 v a y^2 x + u a x^2 y + \frac{1}{3} v a y^3 + 2 v H \]

\[ ii) \quad C_2 = p (x y^2 - y x y - \frac{1}{4} A y^2 x - a x^2 y^2 - \frac{1}{4} a y^4) + 2 v H \]

Let us now apply the general approach in order to find for the system (26) a second invariant of the form:

\[ C = f_0 \dot{y}^4 + f_1 \dot{y}^3 \dot{x} + g_0 \dot{y}^2 + g_1 \dot{x} \dot{y} + h \]

where \( f_i(x, y), \quad i = 0, 1, \quad g_j(x, y), \quad j = 0, 1 \) and \( h(x, y) \) should be determined.

After explicit computations we obtain that the general bi-dimensional Hénon-Heiles system (26) has the integrable form:

\[ H = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 + \frac{A}{2} x^2 + \frac{A}{32} y^2 + a x y^2 + \frac{16 a}{3} x^3 \]

and the following second invariant with quartic velocities:

\[ C = \beta y^4 + 2 \beta y^2 (B + 2a x) \dot{y}^2 - \frac{4}{3} \beta a y^3 \dot{y} - \frac{4}{3} \beta a y^4 (B x - a x^2) + \\
+256 \beta B^2 y^4 - \frac{2}{9} \beta a^2 y^6, \]

where \( \beta, B, a \) are arbitrary constants.

Our cases of integrability (28), (29), (33) coincide with the well known results from [6].

4. CONCLUSIONS

The method we presented is specific for the dynamical systems described by autonomous Hamiltonians but it can be extended for non-autonomous ones [7]. By applying this method for two concrete models, Yang-Mills and Hénon-Heiles, we recovered well-known invariants and, by this, we tested the validity of the method.

Numerical analysis shows an interesting connection between the two models in the integrable cases [8]. Some cases of non-autonomous Hamiltonians will be tackled with in a forthcoming paper [9].
REFERENCES


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