A class of Hamiltonian systems is studied in order to describe, from a mathematical point of view, the configuration of the magnetic field in tokamaks (toroidal devices used in controlled thermonuclear fusion).

General explanations for some experimental observations (concerning the magnetic transport barriers, magnetic reconnection etc) are issued from the analytical properties of the models.

For the systems describing the reversed-shear tokamaks some other specific properties are presented and their influence on the (anomalous) magnetic transport is analyzed.

1. INTRODUCTION

The tokamaks (the abbreviation of the original russian name Torichnaia Kamera Magnitnaia Katushkha) are toroidal chambers in which plasma particles are confined by a magnetic field obtained by the superposition of two basic components acting in the direction of the major, respectively minor curvatures of the torus.

The magnetic field line equations generate Hamiltonian systems [2–5]. In the ideal (integrable) case the helical magnetic field lines lay on constant pressure surfaces (called magnetic surfaces) with topology of nested tori surrounding the magnetic axis (the axis of the torus) [1]. Usually magnetic perturbations appear in experiments because of internal factors (instabilities, fluctuations) or external causes (imperfections, divertor coils). The perturbed systems, generically non integrable [4], are characterized by a complex mixture of regular (periodic or quasiperiodic) and chaotic magnetic field lines.

We are particularly interested in the study of the magnetic transport barriers (TB). From the mathematical point of view, a magnetic transport barrier is an invariant set separating the central stochastic zone, situated near the magnetic axis of the tokamak, from the peripheral stochastic zone, situated near the tokamak’s wall. It cannot be traversed by the magnetic field lines i.e., the magnetic transport through TB is suppressed and the magnetic field lines passing
through the central stochastic zone are confined. From a practical point of view, the study of magnetic transport barriers relates to the more general study of particle internal transport barriers which is of great importance for nuclear fusion because the particle confinement time is increased (the reduction of particle transport is expected due to a reduction of magnetic field line radial wandering and to the fact that charge particle guiding centers follow magnetic field lines).

The magnetic transport barrier cannot be directly observed in experiments but the influence of the magnetic field configuration (through its $q$-profile) on the appearance of transport barriers for particles was reported in experiments in Tore Supra, JT-60U, JET, TFTR and other tokamaks [6–8]. We focused on two important features:

– generally the transport barriers are obtained in presence of a reversed magnetic shear i.e., the $q$-profile has a local maximum near the magnetic axis and a local minimum at a normalized radius between 0.3 and 0.4. The transport barrier appears in the negative shear region or in the low shear region (near the points where $q$ has the minimum value) where a reduced heat diffusivity is observed [6], [9]; in [13] are presented two spatially separated transport barriers located in both the positive and negative shear regions.

– zones with reduced transport can be noticed near the points where $q$ has low order rational values ($q = 1, q = 2, q = 2.5, q = 3$) [10–13].

We obtain here theoretical explanations for the previous experimental observations by studying the transport barriers in a purely magnetic description, using the Rev-Tokamap model [14].

The paper is structured as follows: the Rev-Tokamap model is presented in Section 2; the analytical properties of the system will be used in order to define the transport barrier; the transport barrier will be described and located for various quadratic $q$-profiles and various perturbations in Section 3; Section 4 presents the conclusions of our study.

2. THE REV-TOKAMAP MODEL

The toroidal coordinates $(r, \theta, \zeta)$ are used in order to describe the magnetic line configuration. $\zeta$ is the toroidal angle around the symmetry axis and $(r, \theta)$ are the polar coordinates in a (circular) poloidal cross-section at distance $R_0$ to the symmetry axis and having maximal radius $a$. However, since canonicity of the coordinates is needed for the derivation of the Hamiltonian discrete dynamical system describing the behaviour of the magnetic field lines one uses the toroidal flux $\psi = r^2/2$ instead of the radial coordinate $r$ [15].
Non linear dynamics of the magnetic field in tokamak

3

system, generically nonintegrable, can be studied using the Poincaré map associated to the poloidal section $\mathcal{S}$: $\zeta = \text{cst}$. The intersection of the magnetic field line starting from $(\theta_0, \psi_0)$ with $\mathcal{S}$ after $n$ toroidal rounds will be denoted by $(\theta_n, \psi_n)$. The third coordinate $\zeta$ does not appear here as it is a constant parameter depending only on the position of the poloidal section. The Poincaré map

$$T_K : [0,1) \times \mathbb{R}_+ \rightarrow [0,1) \times \mathbb{R}_+$$
given by $T_K(\theta, \psi) = (\theta_{n+1}, \psi_{n+1})$ (1)

has to be an area-preserving map compatible with the toroidal geometry (if $\psi_0 = 0$ then $\psi_n = 0$ for all $n \in \mathbb{N}$ and if $\psi_0 > 0$ then $\psi_n > 0$ for all $n \in \mathbb{N}$).

A map satisfying the imposed constraints is derived from the mixed generating function [15, 16], namely

$$F_K : [0,1) \times \mathbb{R}_+ \rightarrow [0,1) \times \mathbb{R}_+$$

$$F_K(\theta, \psi) = \psi_{n+1} \theta_n + \alpha_0(\psi_{n+1}) + K \cdot P(\theta, \psi_{n+1}).$$

The corresponding discrete system is

$$\begin{cases}
\psi_{n+1} = \psi_n - K \frac{\partial P(\theta, \psi_{n+1})}{\partial \theta}

\theta_{n+1} = \theta_n + W(\psi_{n+1}) + K \frac{\partial P(\theta, \psi_{n+1})}{\partial \psi_{n+1}} \pmod{1}
\end{cases} \quad (2)
$$

The analytical expression of $T_K$ ($T_K(\theta, \psi)$ is denoted by $(\theta, \psi)$) is obtained by determining $\theta_{n+1}$ and $\psi_{n+1}$ from (2) as a function of $\theta_n$ and $\psi_n$.

The application $W$ is called the winding function and $q \equiv \frac{1}{W}$ is the safety factor, also called the $q$-profile. The magnetic shear is defined by $s = \frac{d \ln q}{d \ln \psi}$. It is positive when $q$ is an increasing function and it is negative when $q$ is a decreasing function.

In order to obtain realistic models of magnetic field, lines R. Balescu proposed the Tokamap model [15] and the Rev-Tokamap model [14] involving a monotonous, respectively a non-monotonous winding function. The perturbation $P = P(\theta, \psi)$ was chosen in order to respect the toroidal geometry.

The Tokamap was obtained by choosing in (2)

$$P(\theta, \psi) = -\frac{1}{(2\pi)^2 \psi} \cos(2\pi \theta) \quad \text{and} \quad W(\psi) = \frac{1}{4}(2 - \psi)(2 - 2\psi + \psi^2).$$

It is a twist map (i.e., $\frac{\partial \theta}{\partial \psi}(\theta, \psi) \neq 0$ for all $(\theta, \psi) \in [0,1) \times \mathbb{R}_+$) compatible with the toroidal geometry. The realistic (decreasing) winding number $W$, in fact the safety factor $q = \frac{1}{W}$ was derived in [17].
An extensive study of the dynamical properties of the Tokamap was performed in [17, 18].

In order to study the reversed shear configuration in [14] it was considered

\[ W(\psi) = w \left[ 1 - A \cdot (C \cdot \psi - 1)^2 \right] \]  

(3)

where

\( A = \frac{w - w_0}{w}, \quad C = 1 + \sqrt{\frac{w - w_1}{w - w_0}} \)

and the same perturbation \( P \) as in the Tokamap model.

The map which generates the system is the Rev-Tokamap namely

\[ T_K : [0, 1) \times \mathbb{R}_+ \rightarrow [0, 1) \times \mathbb{R}_+, \quad T_K (0, \psi) = (\bar{\theta}, \bar{\psi}) \]  

with

\[
\begin{align*}
\bar{\psi} &= \psi - 1 - K \sin(2\pi \theta) + \sqrt{(\psi - 1 - K \sin(2\pi \theta))^2 + 4\psi} \\
\bar{\theta} &= \theta + w \left[ 1 - A \cdot (C \cdot \psi - 1)^2 \right] - \frac{K}{4\pi} \frac{1}{(1 + \psi)^2} \cos(2\pi \theta) \mod 1
\end{align*}
\]

(4)

The winding function \( W \) has a local minimum in \( \psi = 0 \) and a local maximum when \( \psi = 1/C \), hence the (unperturbed) safety factor \( q \) has a local minimum \( q_{min} = \frac{1}{w} \) in \( \psi = 1/C \). The \( q \)-profile corresponding to \( w = 0.67, \ w_0 = 0.3333, \ w_1 = 0.1667 \) is presented in Fig. 1a. \( K \in [0, 2\pi] \) is the stochasticity parameter indicating the amplitude of the perturbation.

The phase portrait of rev-tokamap \((K = 4.5, \ w = 0.67, \ w_0 = 0.3333, \ w_1 = 0.1667)\) is presented in Fig. 1. A periodic magnetic field line intersects the poloidal section in a finite number of points. The intersection of a quasiperiodic magnetic field line with the poloidal section is a closed curve. The intersections of a chaotic magnetic field line with the poloidal section erratically fill a zone having positive area, called chaotic or stochastic zone. Two invariant chaotic zones can be observed. The bounded chaotic zone is situated near the magnetic axis \( \psi = 0 \). In this region chaotic and regular orbits coexist. The unbounded chaotic zone is situated in the peripheral zone of tokamak, near \( \psi = 1 \). In this zone there are no regular orbits. For this reason it is called a “globally stochastic region [4]. Between them, near the minimum of the \( q \)-profile, there is a regular zone formed by periodic orbits and very thin chaotic layers separated by quasiperiodic orbits. This zone can not be traversed by the chaotic orbits. Such a configuration has been shown to be analogous to the appearance of internal transport barriers in reversed shear experiments [4]. The description of this
regular zone is the aim of our study. For this study we will focus on a class of area preserving maps, namely the non-twist maps.

In order to study a general model we will consider area preserving maps (APMs) defined on the annuli \( A = S^1 \times [0, \infty) \) where \( S^1 \) denotes the unit circle.

A prototype of APMs which will be studied in the sequel is \( T_K : A \rightarrow A \)

\[
T_K : \begin{cases}
\bar{\theta} = (\theta + W(\bar{\psi}) + Kg(\theta)h'(\bar{\psi}))(\text{mod}1) \\
\bar{\psi} = \psi - Kg'(\theta)h(\bar{\psi})
\end{cases}
\]

(5)

where \( g : S^1 \rightarrow \mathbb{R} \) and \( h : \mathbb{R} \rightarrow \mathbb{R} \) are \( C^2 \) functions.

The Rev-Tokamap can be obtained by choosing \( W \) from (3) and \( g(\theta) = -\frac{1}{(2\pi)^2}\cos(2\pi\theta), \quad h(\psi) = \frac{\psi}{1+\psi}. \)

The APM \( f : A \rightarrow A, \quad f(\theta, \psi) = (\bar{\theta}, \bar{\psi}) \) is a positive (negative) twist map if there is \( M \) (respectively \( M' \)) such that \( \frac{\partial \bar{\theta}}{\partial \psi}(\theta, \psi) > M > 0 \) respectively \( \frac{\partial \bar{\theta}}{\partial \psi}(\theta, \psi) < M' < 0 \) for every \( (\theta, \psi) \in A \).

The positive (respectively negative) twist maps deviate to the right (respectively to the left) the vertical lines \( \theta = \text{cst}. \)
The inverse of a positive (negative) twist map is a negative (positive) twist map.

The reversed shear magnetic field (non monotonous $W$) can be described by non-twist maps, i.e., maps for which the twist condition is violated. This is the case we will study in this paper. The Rev-Tokamap is a quadratic nontwist area preserving map. Near the magnetic axis $\psi = 0$, the vertical line is deviated to the right (the map is positive-twist) and in the peripheral part of the poloidal section, near $\psi = 1$ the vertical line is tilted to the left (the map is negative-twist). In the positive twist region the magnetic shear is negative because the safety factor is a decreasing function and in the negative twist region the magnetic shear is positive because the safety factor is increasing. The null shear corresponds to the minimum point of the safety factor. The twist condition is violated in the points of the curve $C_1 : \frac{\partial \theta}{\partial \psi}(\theta, \psi) = 0$ which is called the critical twist circle.

For the maps given by (5) the implicit equation of the critical twist circle is

$$C_1 : W'(\psi) + kg(\bar{\theta}) h^*(\psi) = 0.$$  

The critical twist circle of $T_K^{-1}$ has the equation $C_{-1} : \frac{\partial \theta}{\partial \psi}(T_K^{-1}(\theta, \psi)) = 0$ where $T_K^{-1}(\theta, \psi) = (\tilde{\theta}, \tilde{\psi})$. For the maps given by (5) its implicit equation is

$$C_{-1} : W'(\psi) + kg(\bar{\theta}) h^*(\psi) = 0.$$  

One can observe that $(\theta, \psi)$ belongs to $C_1$ if and only if $T_K(\theta, \psi)$ belongs to $C_{-1}$. It results that $C_{-1} = f(C_1)$.

The two critical twist circles $C_1$ and $C_{-1}$ of a quadratic nontwist APM intersect at least in two points. In the case of Rev-Tokamap they intersect exactly in two points.

In the region bounded by the critical twist circles, the maps $T_K$ and $T_K^{-1}$ do not have opposite twist property and the points in these regions have special behaviour.

The regions bounded by $C_1$ or $C_{-1}$ are not invariant (because $C_1$ and $C_{-1}$ are not invariant rotational circles). In order to obtain an invariant set containing $C_1$ and $C_{-1}$ we define the non-twist annulus.

The non-twist annulus (NTA) of a quadratic nontwist APM $f : \mathcal{A} \to \mathcal{A}$, $f(\theta, \psi) = (\bar{\theta}, \bar{\psi})$ is the closure of the orbits of the points $(\theta, \psi) \in \mathcal{A}$ for which $\frac{\partial \theta}{\partial \psi}(\theta, \psi) = 0$. 

NTA collects all the points which have nontwist dynamics. It separates the positive and the negative twist region in the phase space.

In the positive (negative) twist region the map $f$ acts (separately) as a positive (negative) twist map.

The specific non-twist properties (reconnection, the existence of the shearless curve) of $f$ occur in NTA.

In many tokamak experiments with shear-reversed configuration and in numerical simulations concerning nontwist systems a large regular zone was observed in the region where the shear is small, close to 0. This regular zone surrounds the shearless curve and acts as a transport barrier, separating a chaotic zone situated near the magnetic axis (respectively the circle $\psi = 0$ in the Rev-Tokamap model) and the globally stochastic region situated in the tokamak’s peripheral zone (respectively containing points $(\theta, \psi)$ with large $\psi$). We will call it a magnetic internal transport barrier (ITB).

In order to explain the regular behaviour of a nontwist map (5) near the shearless curve we define the regular curve $C_{\text{reg}}: W'(\psi) = 0$. It can be proved that the region where the nontwist map (5) is closest (in $C^0$ topology) to a rigid rotation is the annulus generated by $C_{\text{reg}}$, called the regular annulus. In the regular annulus the map (5) is almost integrable, hence it has regular dynamics in this invariant region. Numerical simulations show that, even for large values of the stochasticity parameter $K$, the critical twist circle $C_1$ and the regular curve $C_{\text{reg}}$ are very closed. The non-twist annulus and the regular annulus practically coincide. It results that the map (5) is almost integrable in the non-twist annulus. The nontwist annulus contains the shearless curve, hence the map (5) is almost integrable in a region containing the shearless curve.

As long as it contains quasiperiodic orbits, NTA acts as a transport barrier i.e., it can not be traversed by the chaotic orbits starting from the positive or from the negative twist region.

3. THE DESCRIPTION OF ITB FOR VARIOUS VALUES OF THE PARAMETERS

3.1. FIXED $q$-PROFILES, VARIOUS VALUES OF $K$

For $K = 0$ the system is integrable hence ITB is the whole phase space $S^1 \times \mathbb{R}_+$ and NTA reduces to the circle $\psi = 0$. By increasing the stochasticity parameter some chaotic orbits appear in the negative and in the positive twist regions the places where Rev-Tokamap is a twist application. The mechanism of destruction of the invariant circles is well-known from the theory of twist maps. The chaotic layers are very thin, practically invisible, in the positive twist region for $K < 2.7$. The bounded chaotic zone develops around the hyperbolic periodic
points of 1/2-type because its stable and unstable manifolds intersect transversally.

In the same time NTA (the nontwist annulus) expends hence the non-twist properties of the system become more important. In Fig. 2 the position of the upper bound of ITB and the bounds of NTA are plotted for the same winding function \((w_0 = 0.3333, w_1 = 0.1667\) and \(w = 0.67\)) and various values of \(K\).

Very sharp changes in the position of the upper bound of ITB can be observed in Fig. 2 when \(K \in (1.2, 1.3)\), \(K \in (1.7, 1.8)\), \(K \in (2.7, 2.8)\) respectively \(K \in (3.4, 3.5)\). This is explained by the chaotisation of some Poincaré-Birchoff chains in the negative twist region. The Poincaré-Birkhoff chain of 1/3-type enters in the globally stochastic zone for a \(K \in (1.2, 1.3)\). The same phenomenon occurs for the periodic orbit of 1/2-type and a \(K \in (1.7, 1.8)\), respectively for the periodic orbit of 4/7-type and a \(K \in (2.7, 2.8)\) or for the periodic orbit of 3/5-type and a \(K \in (3.3, 3.4)\). The width of the islands in a Poincaré-Birchhoff chain decreases when the period increases; for this reason the effect of the chaotisation of 1/2-type periodic orbits \((K \in (1.7, 1.8))\) on the position of ITB’s upper bound is more important than in the other cases.

The rotation numbers of the orbits which exit from ITB \(\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}\right)\) are terms in Farrey tree.

![Fig. 2. – The position of ITB’s upper bound and of NTA’s bounds on the line \(\theta = 0.5\) \((w_0 = 0.3333, w_1 = 0.1667, w = 0.67\) and various values of \(K\).)](image-url)
Many other periodic orbits come in the globally stochastic zone, but this cannot be observed in Fig. 2 because their periods are large and the corresponding island chains are very thin.

NTA is included in ITB for $K < 3.923916$. The upper bound of ITB and the upper bound of NTA coincide for $K = 3.923916$. For $K > 3.923916$ in the upper part of NTA some chaotic orbits are formed hence ITB becomes inside in NTA.

The threshold for breaking up ITB was numerically found to be $K_c = 6.21$. For $K > 6.21$ the chaotic orbits are unbounded and the magnetic confinement is compromised.

3.2. THE SAME VALUE OF $K$ AND VARIOUS WINDING FUNCTIONS

Two families of winding functions will be considered in order to analyse influence of the safety factor on the position and on the width of NTA: winding functions with the same shape and various values on the magnetic axis and winding functions with various shapes and fixed values on the magnetic axis and on the peripheral circle $\psi = 1$.

The winding functions with the same shape were obtained by considering $W = W_0 + a$. In Fig. 3a some winding function in the family are presented.

Fig. 3. – a) Translated winding functions for the Rev-Tokamap. b) the position of NTA for $K = 2.5$ and translated $q$-profiles.
By this change the larger value of the unperturbed winding function $W$ is drastically modified.

The position of the boundaries of NTA on the line $\theta = 0.5$ is presented in Fig. 3b ($K = 2.5$, $W_0$ determined by $w = 0.4$, $w_0 = 0.3333$, $w_1 = 0.1667$ and various $a \in [0, 0.4]$). The enlargement of NTA can be observed when $w$ is closed but larger than a low order rational (1/2, 2/3, 3/4) i.e. $q_{\min}$ is closed but less than the same rational.

The winding functions with various shapes can be obtained by fixing their values $w_0$ and $w_1$ on the magnetic axis $\psi = 0$ respectively on the peripheral circle $\psi = 1$ and by modifying the maximum value $w$.

In Fig. 4a some winding functions in the family are plotted. Fig. 4b is analogous with Fig. 3b ($K = 2.5$, $w_0 = 0.3333$, $w_1 = 0.1667$). The same enlargement of NTA near the low order rational can be reported. It is explained by reconnection arguments: before reconnection the twin island chains involved in the reconnection process enter the non-twist annulus (because the reconnection is a non-twist phenomenon and the non-twist annulus collects all the points with non-twist dynamics) and the non-twist annulus becomes larger. The reconnection of Poincaré-Birchoff chains having the rotation number $\frac{n}{m}$ occurs

![Fig. 4. – a) Winding functions with various shapes, b) The position of NTA’s boundaries on the line $\theta = 0.5$.](image-url)
when the maximum value of $W$, namely $w$ is close but larger than $\frac{n}{m}$. The cases $\frac{n}{m} = \frac{1}{2}, \frac{n}{m} = \frac{2}{3}$ can be observed in Fig. 3 and Fig. 4. After reconnection a double collision-annihilation process occurs [19] and the non-twist annulus becomes thinner.

This phenomenon can be considered responsible for the reduced transport noticed near the points where $q$ has low order rational values [10–13] because a large magnetic transport barrier is more difficultly traversed by the charged particles.

Our study shows that the shape of the winding function has only quantitative effects on the width of NTA. Responsible for the enlargement of NTA is the maximum value of $W$ which corresponds to $q_{\text{min}}$, the minimum value of the safety factor.

4. CONCLUSIONS

In order to study the structure of the magnetic field in tokamak a discrete Hamiltonian system, generated by the Rev-Tokamap was used.

The main dynamical properties of the system were presented and new features related to the transport barrier were pointed out: the non-twist annulus (NTA) was found to be an analytical transport barrier. It was compared with the physical transport barrier for a fixed $q$-profile and increasing perturbations.

The non-twist annuli were located and described also for various $q$-profiles and the influence of the shape and of the minimum value of the safety factor on the position and on the width of NTA was discussed. The analytical results were compared with some experimental observations and a qualitative agreement was pointed out.

Acknowledgement. The author would like to thanks the members of the International working group “Fusion BFR” for their great interest in the present work and in particular R. Balescu, J. Misguich, E. Petrisor and B. Weyssow for many detailed discussions.

REFERENCES

7. X. Garbet, Phys Plasmas 8 (2001), 2793.