In human body, all the conduits are flexible and also collapsible. That is, when the external pressure exceeds the internal pressure, the tube cross-sectional area can be significantly reduced, if not fully diminished. In this paper, a mathematical model describing the fluid dynamics of collapsible tube is presented. Analytical solutions are constructed for the problem using perturbation technique. The computer extension of the resulting power series solutions, its analysis and analytic continuation, including bifurcation study is performed with a special type of Hermite-Padé approximants. We obtained accurately a turning point \( R_c \) (\( R \) the flow Reynolds number), as well as the asymptotic behaviour of the skin friction and fluid pressure gradient as \( R \to 0 \) on the secondary solution branch. The model is most appropriate to simulate wind tunnel tests on rheological phenomenon in physiological systems.

Key words: Collapsible tube, Hermite-Padé approximants, Bifurcation study.

2000 Mathematics Subject Classification: 76E25.

1. INTRODUCTION

Fluid flow through collapsible tubes is a complex problem due to the interaction between the tube-wall and the flowing fluid, Heil (1997). Collapsible tubes are easily deformed by negative transmural pressure owing to marked reduction of rigidity. Thus, they show a considerable nonlinearity and reveal various complicated phenomena. It is usually used to simulate biological flows such as blood flow in arteries or veins, flow of urine in urethras and airflow in the bronchial airways. These investigations are very useful for the study and prediction of many diseases, as the lung disease (asthma and emphysema), or the cardiovascular diseases (heart stroke). The major research goal remains, the full understanding of the flow structure and the mechanism driving this flow. Many previous theoretical works on flow in collapsible tubes concentrated on the development and analysis of simpler models, by reducing the spatial dimension of the problem, which involve a number of ad-hoc assumptions e.g., Contrad (1969), Grotberg (1971), Flaherty et al. (1972), Cowley (1982), Bonis & Ribreau (1987), etc. Experimental example of the work that have been done on
collapsible tube includes the one performed with finite-length elastic tubes whose upstream and downstream ends are held open (i.e., Starling-resistor, Brower & Scholten 1975, Bertram 1986). Inside a pressure chamber, thin-walled elastic tube (made of latex rubber) is mounted on two rigid tubes. Fluid (liquid or gas) typically water or air respectively is driven through the tube, either by applying a controlled pressure-drop between the ends of the rigid tubes or by controlling the flow rate. If the external pressure exceeds the fluid pressure by a sufficiently large amount, the tube buckles non-axisymmetrically, which then leads to a nonlinear relation between pressure-drop and flow rate. At sufficiently large Reynolds numbers, the system produces self-excited oscillations, and exhibits hysteresis in transitions between dynamical states, multiple modes of oscillations (each having distinct frequency range), rich and complex nonlinear dynamics (Bertram et al. 1990). The inertia and resistance of the fluid in the supporting rigid tubes have an important influence on the system’s overall dynamics. This experiment forms the basis for most recent theoretical investigations due to its three-dimensional nature. Meanwhile, Bertram and Pedley (1982), Bertram and Raymond (1991) investigated two-dimensional channel theoretical model with one wall of the channel been replaced by a membrane under longitudinal tension, viscous flow is driven through the channel by an imposed pressure-drop. The variation between the external pressure and the internal flow determine the deformation of the membrane. The dynamics of the problem is described by nonlinear ODE’s whose numerical solutions exhibit oscillatory behaviour reminiscent of that observed in experiments. Despite the difficulties of producing two-dimensional flows experimentally, this system still attracted considerable theoretical attentions. Since it avoid the complications of three-dimensional flows found in the Starling-resistor, while still exhibiting phenomena such as flow limitation and self-excited oscillations.

Meanwhile, mathematical model of physical phenomena often results in nonlinear equations for some unknown function. Usually the problems cannot be solved exactly. The solutions of these nonlinear systems are dominated by their singularities: physically, a real singularity controls the local behaviour of a solution. There is a long tradition in applied mathematics to solve nonlinear problems by expansion in powers of some “small” perturbation parameter. The advantage of this approach is that it reduces the original nonlinear problem to a sequence of linear problems (Makinde, 1999). However, it is not always possible to find an unlimited number of terms of power series. Often it is possible to obtain a finite number of terms of that series and these may contain a remarkable amount of information. One can reveal the solution behaviour near the critical points by analysing partial sum (Makinde, 2001). Over the last quarter century, highly specialised techniques have been developed to improved the series summation and also used to extract the required information of the singularities.
from a finite number of series coefficients. The most frequently used methods include Domb-Sykes (1957), Shafer (1974), Hunter and Guerrieri (1980), Sergeev (1986), Drazin and Tourigny (1996), Sergeev and Goodson (1998), Makinde et al. (2002), etc.

In this paper, we investigate the flow of a viscous incompressible fluid in a collapsible tube. A special type of Hermite-Padé approximants technique is presented and utilised to analyse the flow structure. The chief merit of this new method is its ability to reveal the dominant singularity in the flow field together with solution branches of the underlying problem in addition to the one represented by the original series. In Sections 2 and 3, we establish the mathematical formulation for the problem. With the benefit of twenty years of advances in computing hardware, we are able to find many terms of the solution series as presented in Section 4. The methods used to sum the series are described in Section 5. In Section 6, we discuss the pertinent results.

2. MATHEMATICAL FORMULATION

The problem under consideration is that of transient flow of a viscous incompressible fluid in a collapsible tube. Take a cylindrical polar coordinate system \((r, \theta, z)\) where \(0z\) lies along the centre of the tube, \(r\) is the distance measured radially and \(\theta\) is the azimuthal angle. Let \(u\) and \(v\) be the velocity components in the directions of \(z\) and \(r\) increasing respectively. It is assume that the tube’s wall is at \(r = a_0 \sqrt{1 - \alpha t}\), where \(\alpha\) is a constant of dimension \([T^{-1}]\) which characterizes unsteadiness in the flow field, \(a_0\) is the characteristic radius of the tube at time \(t = 0\) as shown in figure below.

![Fig. 2.1. – Schematic diagram of the problem.](image)

Then, for axisymmetric unsteady viscous incompressible flow, the Navier-Stokes equations are
\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = -\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \nu \nabla^2 \mathbf{u},
\]
(2.01)

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{v} + \nabla P = -\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \nu \left( \nabla^2 \mathbf{v} - \frac{\mathbf{v}}{r^2} \right),
\]
(2.02)

where \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \), \( P \) is the pressure, \( \rho \) the density and \( \nu \) the kinematic viscosity of the fluid. The equation of continuity is

\[
\frac{\partial \rho}{\partial t} + r \frac{\partial \rho}{\partial r} = 0.
\]
(2.03)

The appropriate boundary conditions are:

1. Regularity of solution along \( z \)-axis, \textit{i.e.},

\[
\frac{\partial \mathbf{u}}{\partial r} = 0, \quad v = 0, \quad \text{on} \quad r = 0.
\]
(2.04)

2. The axial and normal velocities at the wall are prescribed as:

\[
u = 0, \quad v = \frac{da}{dt}, \quad \text{on} \quad r = a(t).
\]
(2.05)

We introduce the stream-function \( \Psi \) and vorticity \( \omega \) in the following manner:

\[
u = \frac{1}{r} \frac{\partial \Psi}{\partial r} \quad \text{and} \quad \omega = -\frac{1}{r^2} \frac{\partial \Psi}{\partial z},
\]
(2.06)

\[
u = \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \Psi}{\partial r},
\]
(2.07)

Eliminating pressure \( P \) from (2.01) and (2.02) by using (2.06) and (2.07) we get:

\[
\frac{\partial \omega}{\partial t} + \frac{1}{r} \frac{\partial (\Psi, \omega)}{\partial r} + \frac{\omega}{r^2} \frac{\partial \Psi}{\partial z} = \nu \left[ \nabla^2 \omega - \frac{\omega}{r^2} \right], \quad \omega = -\nabla^2 \Psi,
\]
(2.08)

\[
\frac{\partial \Psi}{\partial r} = 0, \quad \frac{\partial \Psi}{\partial z} = -a \frac{da}{dt}, \quad \text{on} \quad r = a(t),
\]
(2.09)

\[
\frac{\partial \Psi}{\partial r} \left( \frac{1}{r} \frac{\partial \Psi}{\partial r} \right) = 0, \quad \frac{\partial \Psi}{\partial z} = 0, \quad \text{on} \quad r = 0.
\]
(2.10)

We introduce the following transformations:

\[
\eta = \frac{r}{a_0 \sqrt{1 - \alpha t}}, \quad \Psi = \frac{a_0^2 \alpha F(\eta)}{2}, \quad \omega = -\frac{\alpha G(\eta)}{2a_0 \left( \sqrt{1 - \alpha t} \right)^3}.
\]
(2.11)
Substituting equation (2.11) into equations (2.08)–(2.10), we obtain:

$$\frac{d}{d\eta} \left[ \frac{1}{\eta} \frac{d(\eta G)}{d\eta} \right] = R \left[ \frac{G}{\eta} \frac{dF}{d\eta} - F \frac{d}{d\eta} \left( \frac{G}{\eta} \right) + \eta \frac{dG}{d\eta} + 3G \right], \quad G = \frac{d}{d\eta} \left( \frac{1}{\eta} \frac{dF}{d\eta} \right), \quad (2.12)$$

$$\frac{dF}{d\eta} = 0, \quad F = 1, \quad \text{on} \quad \eta = 1, \quad (2.13)$$

$$\frac{d}{d\eta} \left( \frac{1}{\eta} \frac{dF}{d\eta} \right) = 0, \quad F = 0, \quad \text{on} \quad \eta = 0, \quad (2.14)$$

where $R = a_0^2 \alpha / 2 \nu$ is the local Reynolds number ($R > 0$ represents contraction while $R < 0$ represents expansion of the tube’s wall).

### 3. METHOD OF SOLUTION

To solve equations (2.12)–(2.14), it is convenient to take a power series expansion in the flow local Reynolds number $R$, *i.e.*, 

$$F = \sum_{i=0}^{\infty} F_i R^i, \quad G = \sum_{i=0}^{\infty} G_i R^i, \quad (3.01)$$

Substitute (3.01) into equations (2.12)–(2.14) and collecting the coefficients of like powers of $R$, we obtain the following:

**Zeroth Order:**

$$\frac{d}{d\eta} \left( \frac{1}{\eta} \frac{d(\eta G_0)}{d\eta} \right) = 0, \quad G_0 = \frac{d}{d\eta} \left( \frac{1}{\eta} \frac{dF_0}{d\eta} \right), \quad (3.02)$$

$$\frac{dF_0}{d\eta} = 0, \quad F_0 = 1, \quad \text{on} \quad \eta = 1, \quad (3.03)$$

$$\frac{d}{d\eta} \left( \frac{1}{\eta} \frac{dF_0}{d\eta} \right) = 0, \quad F_0 = 0, \quad \text{on} \quad \eta = 0, \quad (3.04)$$

**Higher Order ($n \geq 1$):**

$$\frac{d}{d\eta} \left( \frac{1}{\eta} \frac{d(\eta G_n)}{d\eta} \right) = R \left[ \sum_{i=0}^{n-1} G_i \frac{dF_{n-i-1}}{d\eta} - F_i \frac{d}{d\eta} \left( \frac{G_{n-i-1}}{\eta} \right) \right] + \eta \frac{dG_{n-1}}{d\eta} + 3G_{n-1}, \quad (3.05)$$

$$G_n = \frac{d}{d\eta} \left( \frac{1}{\eta} \frac{dF_n}{d\eta} \right),$$
\[
\frac{dF_n}{d\eta} = 0, \quad F_n = 0, \quad \text{on} \quad \eta = 1, \quad (3.06)
\]
\[
\frac{d}{d\eta} \left( \frac{1}{\eta} \frac{dF_n}{d\eta} \right) = 0, \quad F_n = 0, \quad \text{on} \quad \eta = 0. \quad (3.07)
\]

We have written a MAPLE program that calculates successively the coefficients of the solution series. In outline, it consists of the following segments:

1. Declaration of arrays for the solution series coefficients \( e.g., \ F = \text{array}(0..43), \ G = \text{array}(0..43) \).
2. Input the leading order term and their derivatives \( i.e., \ F_0, \ G_0 \).
3. Using a MAPLE loop procedure, iterate to solve equations (3.05)–(3.07) for the higher order terms \( i.e., \ F_n, \ G_n, \ n = 1, 2, 3, \ldots \).
4. Compute the skin friction and axial pressure gradient coefficients.

Details of the MAPLE program can be found in the appendix. Some of the solution stream-function and vorticity are then given as follows:

\[
F(\eta) = (2\eta^2 - \eta^4) + \frac{R_\eta^2}{36} (\eta^2 - 1)^2 (\eta^2 - 10) + \frac{R^2 \eta^2}{1940400} (\eta^2 - 1)^2 (2\eta^6 - 101\eta^4 + 596\eta^2 - 1057) + O(R^3)
\]
\[
G(\eta) = -8\eta + \frac{2Rr}{3} (2\eta^4 - 12\eta^3 + 7) - \frac{R^2 \eta}{135} (3\eta^8 - 105\eta^6 + 480\eta^4 - 705\eta^2 + 271) + O(R^3)
\]

The wall skin friction is given by

\[
\tau_w = -\mu \frac{\partial \mu}{\partial \tau} = -\frac{\alpha \mu z}{2a_0 \sqrt{1 - \alpha \tau}} \frac{d}{d\eta} \left( \frac{1}{\eta} \frac{dF}{d\eta} \right) \quad \text{at} \quad \eta = 1.
\]

where \( \mu \) is the dynamic coefficient of viscosity. From the axial component of the Navier-Stokes equations, the pressure drop in the longitudinal direction can be obtained. Let

\[
\frac{\partial P}{\partial z} = \frac{\mu \alpha z A}{2a_0^2 (1 - \alpha \tau)^2},
\]

we substitute (2.11) together with (3.11) into (1.01) and obtain

\[
A = \frac{1}{\eta} \frac{d}{d\eta} \left[ \eta \frac{d}{d\eta} \left( \frac{1}{\eta} \frac{dF}{d\eta} \right) \right] - R \left( \frac{1}{\eta} \frac{dF}{d\eta} \right)^2 - \frac{F}{\eta} \frac{d}{d\eta} \left( \frac{1}{\eta} \frac{dF}{d\eta} \right) + \eta \frac{d}{d\eta} \left( \frac{1}{\eta} \frac{dF}{d\eta} \right) + 2 \frac{dF}{d\eta}.
\]
In order to investigate the flow structure at moderately large Reynolds numbers, we expand $H = -G(\eta)$ at $\eta = 1$ and $A$ representing wall skin friction and axial pressure gradient parameters respectively, in powers of the Reynolds number $R$ i.e.,

$$H = 8 + 2R - \frac{56R^2}{135} + \frac{3389R^3}{22680} - \frac{5104949R^4}{81648000} + \frac{12136339R^5}{143700480} + \ldots \quad (4.01)$$

$$A = 8 + \frac{38R}{3} - \frac{152R^2}{135} + \frac{3287R^3}{7560} - \frac{15420829R^4}{81648000} + \frac{133921837R^5}{1539648000} + \ldots \quad (4.02)$$

We compute the first 44 coefficients of the above series as shown in Table 4.01 below. The signs of the coefficients alternate after the second term and are monotonically decreasing in magnitude. Hence, the convergence of the series may be limited by a singularity on the negative real axis.

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<th>$A[I]$</th>
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<td>8.000000000000000000000</td>
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</tr>
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<tr>
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5. SERIES IMPROVEMENT TECHNIQUE: (A NEW APPROACH)

The main tool of this paper is a simple technique of series summation based on the generalization of Padé approximants and may be described as follows. Let us suppose that the partial sum

$$U_{N-1}(\lambda) = \sum_{i=0}^{N-1} a_i \lambda^i = U(\lambda) + O(\lambda^N) \quad \text{as} \quad \lambda \to 0, \quad (5.01)$$

is given. We are concerned with the bifurcation study by analytic continuation as well as the dominant behaviour of the solution by using partial sum (5.01). We expect that the accuracy of the critical parameters will ensure the accuracy of the
solution. It is well known that the dominant behaviour of a solution of a linear ordinary differential equation can often be written as Guttmann (1989),

$$U(\lambda) \approx \begin{cases} K(\lambda_c - \lambda)^\alpha & \text{for } \alpha \neq 0, 1, 2, \ldots \\ K(\lambda_c - \lambda)^\alpha \ln|\lambda_c - \lambda| & \text{for } \alpha = 0, 1, 2, \ldots \end{cases} \text{ as } \lambda \to \lambda_c \quad (5.02)$$

where $K$ is some constant and $\lambda_c$ is the critical point with the exponent $\alpha$. However, we shall make the simplest hypothesis in the contest of nonlinear problems by assuming the $U(\lambda)$ is the local representation of an algebraic function of $\lambda$. Therefore, we seek an expression of the form

$$F_d(\lambda, U_{N-1}) = A_{0N}(\lambda) + A_{1N}(\lambda)U^{(1)} + A_{2N}(\lambda)U^{(2)} + A_{3N}(\lambda)U^{(3)}, \quad (5.03)$$

such that

$$A_{0N}(\lambda) = 1, \quad A_{1N}(\lambda) = \sum_{j=1}^{d+i} b_{ij} \lambda^{j-1}, \quad (5.04)$$

and

$$F_d(\lambda, U) = O(\lambda^{N+1}) \quad \text{as } \lambda \to 0, \quad (5.05)$$

where $d \geq 1$, $i = 1, 2, 3$. The condition (5.04) normalises the $F_d$ and ensures that the order of series $A_{iN}$ increases as $i$ and $d$ increase in value. There are thus $3(2 + d)$ undetermined coefficients $b_{ij}$ in the expression (5.03). The requirement (5.05) reduces the problem to a system of $N$ linear equations for the unknown coefficients of $F_d$. The entries of the underlying matrix depend only on the $N$ given coefficients $a_i$. Henceforth, we shall take

$$N = 3(2 + d), \quad (5.06)$$

so that the number of equations equals the number of unknowns. Equation (5.05) is a new special type of Hermite-Padé approximants. Both the algebraic and differential approximants form of equation (5.05) are considered. For instance, we let

$$U^{(1)} = U, \quad U^{(2)} = DU, \quad U^{(3)} = D^2U, \quad (5.07)$$

and obtain a cubic Padé approximant. This gives an extension of the idea of quadratic Padé approximants by Shafer (1974) and Sergeev (1986). Furthermore, Sergeev and Goodson (1998), Drazin and Tourigny (1996) had also suggested a similar form of higher order algebraic approximants. Generally, this enables us to obtain solution branches of the underlying problem in addition to the one represented by the original series. In the same manner, we let

$$U^{(1)} = U, \quad U^{(2)} = DU, \quad U^{(3)} = D^2U, \quad (5.08)$$
in equation (5.04), where $D$ is the differential operator given by $D = d/d\lambda$. This leads to a second order differential approximants. It is an extension of the integral approximants idea by Hunter and Baker (1979) and enables us to obtain the dominant singularity in the flow field \textit{i.e.} by equating the coefficient $A_{3N}(\lambda)$ in the equation (5.05) to zero. The critical exponent $\alpha_N$ can easily be found by using Newton’s polygon algorithm. However, it is well known that, in the case of algebraic equations, the only singularities that are structurally stable are simple turning points. Hence, in practice, one almost invariably obtains $\alpha_N = 1/2$. If we assume a singularity of algebraic type as in equation (5.02), then the exponent may be approximated by

$$\alpha_N = 1 - \frac{A_{2N}(\lambda_{CN})}{DA_{3N}(\lambda_{CN})}.$$  \hspace{1cm} (5.09)

Using the above procedure, we performed series summation and improvement study on the solution series obtained in table (4.01). Our results show the dominant singularity in the flow field to be $R_c = -1.6739367347720807$ (which corresponds to the radius of convergence and the turning point in the flow field) with the critical exponent $\alpha_c = 0.5$ as shown in the Table 5.01. We also noticed that $A \sim A_{+1}/R$ and $H \sim H_{+1}/R$ as $R \to 0$ on the secondary solution branch where $A_{+1} \approx -280.98050$ and $H_{+1} \approx -67.6702067$. It is noteworthy to mention that the wall shear stress $H \to 0$ as $R \to -1.6431409627402$, \textit{i.e.}, separation and possible flow reversal occur due to tube wall expansion.

<table>
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<th>$d$</th>
<th>$N$</th>
<th>$R_c$</th>
<th>$A_{+1}$</th>
<th>$H_{+1}$</th>
<th>$\alpha_c$</th>
</tr>
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<tr>
<td>1</td>
<td>9</td>
<td>-1.6863617034262282</td>
<td>-276.48020</td>
<td>67.3394184</td>
<td>0.498634106</td>
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<tr>
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<td>-281.11885</td>
<td>67.6661825</td>
<td>0.498726578</td>
</tr>
<tr>
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<td>-280.98050</td>
<td>67.6702067</td>
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<tr>
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<tr>
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<tr>
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<td>67.6702067</td>
<td>0.500000000</td>
</tr>
</tbody>
</table>

6. GRAPHICAL RESULTS AND DISCUSSION

In Fig. 6.01, we observed that the fluid axial velocity profile is parabolic with maximum value at centerline and minimum at the plates. It is interesting to
note that the fluid axial velocity generally decreases with an increase in tube contraction due to the strong influence of the negative transmural pressure owing to marked reduction of rigidity (i.e. $R > 0$). In reality, this is possible, since contraction brings about a reduction in the tube’s cross-sectional area, hence, decreasing the amount of flow passing through the compressed region. Table 5.01, shows the convergence of the dominant singularity $R_c$ in the flow field together with its corresponding exponent $\alpha_c$, as well as the asymptotic behaviour of wall skin friction and pressure gradient as the flow Reynolds number tends to zero. It is noteworthy also that $R_c$ is the bifurcation point and lies in the negative real axis of the flow Reynolds number $R$, i.e., the region representing tube’s contraction. This critical value of Reynolds number enables the biomedical engineers to determine accurately the maximum expansion of the tube walls due to the variation in the tube’s external and internal pressure i.e., $a_0 = \sqrt{2\nu R_c / \alpha}$. Figs. 6.02 and 6.03 show the sketch of bifurcation diagrams for the skin friction and fluid axial pressure gradient parameters. For tube’s contraction i.e., $R > 0$, only one solution branch exist i.e., type I. This is the primary solution branch and it shows that the wall skin friction and fluid axial pressure gradient increase with increase in $R$. In the expansion region i.e., $R < 0$, two solution branches are identified (i.e., type I, II). A simple turning point with exponent $\alpha_c = 0.5$ exits between type I and type II solution branches i.e., $R_c$. We observed that the type II solution is physically unreasonable but mathematically interesting. It is interesting to note that the turning point here also corresponds to the dominant singularity in the flow field. Finally, in this paper, we have proposed a new form of series summation and improvement technique based on the generalizations of Padé approximants i.e.,
a special type of Hermite-Padé approximant. We have applied this method to investigate the problem of squeezing flow in parallel plates viscometer with great success. The chief novelty of this procedure is its ability to reveal the dominant singularities together with solution branches of the underlying nonlinear problem in addition to the branch represented locally by the original series. Generally, we have found that this new method is very competitive. However, we have not yet develop a theory that would explain its strengths and limitations and so we have relied on intelligent numerical investigation.
APPENDIX

A1: The Maple procedure to solve the equations (3.05) to (3.07).

```maple
# Here we declare the arrays to store the computed results
F:=array(0..34): G:=array(0..34): Fr:=array(0..34): Gr:=array(0..34):
# Here we input the zero order solution F[0] and G[0].
F[0]:=(2*r^2-r^4): G[0]:=-8*r:
Fr[0]:=diff(F[0],r): Gr[0]:=diff(G[0],r):

# This computes the higher order teams i.e. n>0.
for n from 1 by 1 to 43 do
    A1:=normal(1/r*sum(g[i]*Fr[n-i-1]+f[i]*(G[n-i-1]/r-Gr[n-i-1]),i=0..n1)):
    A2:=normal(r*Gr[n-1]+3*g[n-1]):
    A:=R*(A1+A2):
    A1:=0:A2:=0: g11:=normal(r*(int(A,r)+K)):
    A:=0: g1:=normal(int(g11,r)/r):
    g11:=0: f11:=normal(int(f11,r)+M)):
    f1:=normal(int(f11,r)):
r:=1: K:=normal(solve(f11=0,K)):
    M:=normal(solve(f1=0,M)):
r:='r':
    f11:=0: F[n]:=normal(f1):
    f1:=0: G[n]:=normal(g1):
    g1:=0: F[n]:=normal(diff(F[n],r)):
    G[n]:=normal(diff(G[n],r)):
    K:='K': M:='M': print(F[n]): print(G[n]);
end do:
quit();
```

REFERENCES


