The equation of the orbit of a relativistic particle moving in an arbitrary central force field is derived. Straightforward generalizations of well-known first and second order differential equations are thus given. It is pointed out that the relativistic equation of the orbit has the same form as in the nonrelativistic case, the only changes consisting in the appearance of additional terms proportional to $1/c^2$ in both potential and total energies.

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1. INTRODUCTION

Using the equations of motion of a particle in a central force field one obtains easily first and second order differential equations for the orbit of the particle. In classical mechanics textbooks this is done starting from the nonrelativistic equations of motion. Relativistic calculations are usually done only for the particular case of the coulombian force field. In this paper we consider a relativistic particle moving in an arbitrary central force field. We obtain the corresponding first and second order differential equations for its orbit, generalizing well-known equations which are thus put in agreement with the special theory of relativity.

2. RELATIVISTIC CALCULATIONS

We consider a particle that has a radius vector $\vec{r}$ with respect to an inertial frame of reference. The relativistic form of Newton’s second law [1] states that the time derivative of the linear momentum of the particle, $\vec{p}$, is equal to the force exerted on it, $\vec{F}$,

$$\vec{p} = \vec{F},$$

(1)
where \( \vec{p} \) is defined as
\[
\vec{p} = m_0 \gamma \vec{\gamma}.
\] (2)

As usually, \( m_0 \) is the rest mass of the particle, \( \vec{r} \) is its velocity and the typical relativistic factor \( \gamma \) is given by
\[
\gamma = \frac{1}{\sqrt{1 - \vec{r}^2 / c^2}},
\] (3)
c being the speed of light in vacuum.

Calculating the time derivative of \( \gamma \) we obtain
\[
\dot{\gamma} = \frac{1}{m_0 c^2} \vec{p} \cdot \vec{F}.
\] (4)

Consequently eq. (1) can be written in the more convenient form
\[
m_0 \gamma \ddot{\vec{r}} = \vec{F} - \frac{1}{c^2} \left( \vec{r} \cdot \vec{F} \right) \vec{r}.
\] (5)

The kinetic energy of the particle, \( T \), is defined as
\[
T = m_0 c^2 (\gamma - 1)
\] (6)
and its angular momentum, \( \vec{\ell} \), as
\[
\vec{\ell} = \vec{r} \times \vec{p}.
\] (7)

According to the general theorems of mechanics valid in the relativistic case too, we can conclude that if the particle is moving in a central force field \( \vec{F}(\vec{r}) \) described by the potential energy \( U(r) \)
\[
\vec{F}(\vec{r}) = -\frac{dU}{dr} \frac{\vec{r}}{r},
\] (8)
the total energy and the angular momentum are conserved.

Using the spherical coordinates and choosing the polar axis in the direction of the conserved angular momentum, the coordinates \( r \) and \( \phi \) give the position of the particle in the plane of the motion. Thus, the two above-mentioned first integrals of the motion can be written as
\[
m_0 c^2 (\gamma - 1) + U(r) = E,
\] (9)
\[
m_0 \gamma r^2 \phi = \ell,
\] (10)
where the constant values of the total energy and of the angular momentum are \( E \) and \( \ell \), respectively. We also notice that \( \gamma \) is now given by
\[ \gamma = \frac{1}{\sqrt{1 - \frac{\dot{r}^2 + r^2 \dot{\phi}^2}{c^2}}}. \]  
\quad (11)

Following the same procedure as in the nonrelativistic case, to obtain the equation of the orbit we express the time derivatives with the next formula (see eq. (10))

\[ \frac{d}{dt} = \frac{\ell u^2}{m_0 \gamma} \frac{d}{d\phi}, \]

where the more convenient variable \( u = \frac{1}{r} \) is used. For \( \dot{r} \) we get

\[ \dot{r} = -\frac{\ell}{m_0 \gamma} \frac{du}{d\phi}. \]  
\quad (12)

From eq. (9) \( \gamma \) is given by

\[ \gamma = 1 + \frac{E - U(r)}{m_0 c^2} \]  
\quad (13)

and from eqs. (10) and (11), using eq. (12), we have

\[ \gamma^2 = 1 + \frac{\ell^2}{m_0^2 c^2} \left[ \left( \frac{du}{d\phi} \right)^2 + u^2 \right]. \]  
\quad (14)

Eliminating \( \gamma \) between eqs. (13) and (14) we obtain the following first order differential equation for the orbit of the particle

\[ \frac{\ell^2}{2m_0} \left[ \left( \frac{du}{d\phi} \right)^2 + u^2 \right] + \ddot{U}\left( \frac{1}{u} \right) = \ddot{E}, \]  
\quad (15)

where

\[ \ddot{U}\left( \frac{1}{u} \right) = \left( 1 + \frac{E}{m_0 c^2} \right) U\left( \frac{1}{u} \right) - \frac{1}{2m_0 c^2} U\left( \frac{1}{u} \right)^2, \]  
\quad (16)

\[ \ddot{E} = E \left( 1 + \frac{E}{2m_0 c^2} \right). \]  
\quad (17)

Equation (15) is formally identical with the equation of the orbit obtained for a nonrelativistic particle of mass \( m_0 \), angular momentum \( \ell \), energy \( \dot{E} \) moving in a central force field described by the potential energy \( \ddot{U} \). We also notice that in the limiting case \( c \to \infty \) we have \( \ddot{U} \to U \) and \( \ddot{E} \to E \). Therefore, we conclude that the problem of finding the orbit of a relativistic particle having
angular momentum \( \ell \) and energy \( E \) moving in a central force field \( U(r) \) is equivalent to the problem of finding the orbit of the same nonrelativistic particle having the same angular momentum \( \ell \) and a modified energy \( \tilde{E} \) related to \( E \) according to eq. (17), a particle which moves in a modified central force field \( \tilde{U}(r) \) related to \( U(r) \) as eq. (16) states.

In some cases it is easier to integrate the second order differential equation of the orbit. This equation can be derived starting from eq. (5). The following two scalar equations are obtained

\[
m_0 \gamma (2 \dot{\phi} \dot{r} - r \ddot{\phi}) = -\frac{r \dot{r} \dot{\phi}}{c^2} f(r), \tag{18}
\]

\[
m_0 \gamma (\ddot{r} - r \ddot{\phi}) = \left(1 - \frac{\gamma^2}{c^2}\right) f(r), \tag{19}
\]

where \( f(r) = -\frac{dU}{dr} \). Like in the nonrelativistic case, eq. (18) expresses the conservation of the angular momentum as we can easily see by considering the time derivative of eq. (10). In handling eq. (19), we use eqs. (4), (10), (12) and

\[
\dot{r} = -\frac{\ell^2}{m_0 \gamma^2} \left( u^2 \frac{d^2 u}{d\phi^2} + \frac{1}{m_0 c^2 \gamma} \left( \frac{du}{d\phi}\right)^2 f\left(\frac{1}{u}\right) \right). \tag{20}
\]

After a straightforward calculation, eq. (19) can be written in the form of the following second order differential equation for the orbit

\[
\frac{\ell^2}{m_0 \gamma} u^2 \left( \frac{d^2 u}{d\phi^2} + u \right) = -f\left(\frac{1}{u}\right). \tag{21}
\]

When \( c \to \infty \), \( \gamma \) becomes equal to the unity and in this case the equation is well-known (see [2] and eq. (3–34a) in [1]). In French textbooks it is called the Binet equation (see [3]).

Thus, eq. (21) is the generalization of the Binet equation for the case of the relativistic motion of a particle in a central force field. A remarkable fact is that the only change that is to be made in the nonrelativistic equation in order to obtain its relativistic counterpart is the mere replacement of the rest mass of the particle, \( m_0 \), with \( m_0 \gamma \). This is due to a cancellation occurring in eq. (19). Indeed, the second term in the expression of \( \ddot{r} \) given by eq. (20) is equal to the second term in the rhs of eq. (19) (see also eq. (12)).

Inserting the dependence of \( \gamma \) upon \( u \) as given by eq. (13) and using the formula

\[
f\left(\frac{1}{u}\right) = u^2 \frac{d}{du} U\left(\frac{1}{u}\right),
\]
Relativistic equation of the orbit of a particle  

Eq. (21) takes the form

$$\frac{\ell^2}{m_0} \left( \frac{d^2 u}{dq^2} + u \right) = - \frac{d}{du} \tilde{U} \left( \frac{1}{u} \right),$$  \hspace{1cm} (22)$$

where $\tilde{U} \left( \frac{1}{u} \right)$ is given by eq. (16). A check of this result is done noticing that it can also be obtained by a simple differentiation of eq. (15) with respect to $u$.

3. CONCLUSIONS

The main result of this work, that was already mentioned in connection with eq. (15), is that the equation of the orbit of a relativistic particle, i.e., eq. (22), is formally identical with the equation of the orbit of a nonrelativistic particle moving in a central force field described by the potential energy $U(r)$ (see eq. (3–34b) in [1]). Thus, we conclude that the problem of finding the orbit of a relativistic particle in a central force field $U(r)$ is reduced to the problem of finding the orbit of the same nonrelativistic particle in a modified central force field $\tilde{U}(r)$, related to $U(r)$ as seen in eq. (16). We notice that the value of the total energy of the particle, $E$, enters as a parameter in $\tilde{U}$. Consequently, in some minor aspects, the calculations involved differ from the nonrelativistic ones, but the basic procedures developed in the classical mechanics textbooks remain valid, such as the use of eq. (15) to find the so-called classically allowed domains of the motion, the turning points, etc. In some cases, such as the coulombian one, eq. (22) is useful for a more rapid calculation of the orbit. Thus, equations (15) and (22) are useful starting points for an exhaustive study of the possible orbits of relativistic particles moving in arbitrary central force fields.

REFERENCES