The aim of this paper is to study the SO(3, 1) U(1) gauge minimally coupled charged spinless field to a spherically symmetric curved space-time. It is derived the first order analytically approximation solution for the system of Klein-Gordon-Maxwell-Einstein equations. Using these solutions, it is evaluated the electric current’s components and further the boson system electric charge. The considered space–time manifold generalize an anterior one and the chosen metric tensor is of McCrea type.

Key word: Klein-Gordon equation, Maxwell equation, Einstein equation.

1. INTRODUCTION

The study of boson stars (BS) started with the work of Kaup [1] and Ruffini and Bonazzalo [2], who found, for spherical symmetric equilibrium, asymptotical solutions of the Einstein-Klein-Gordon equations. After the mid 1980’s, a major interest has been focused on macroscopic stable boson stars, since they have considered to provide a considerable fraction of the non-baryonic part of dark matter [3]. These configurations are “macroscopic quantum states” and are prevented from collapsing gravitationally only by the Heisenberg uncertainty principle.

The presence of dark matter has been established indirectly in a wide range of scale of the universe, from that of individual galaxies to the entire universe itself. Though direct measurements of the nature of the dark matter have not yielded any result, speculations on its composition vary from baryonic to non-baryonic matter.

Non-interacting complex scalar fields [1, 2] were originally considered for the constituents composing boson stars. The problem has been successfully worked out in low dimensional gravity [4], while in four dimensions, fields interacting via gravity have been investigated mainly by numerical calculations [5].
In present paper we derived an evaluation of the electric charge for a symmetric McCrea space-time. This curved manifold generalizes the anterior studied space-time [6].

2. FIELDS EQUATIONS ON CURVED SPACE-TIME

Let us consider a spherical symmetric configuration described by a metric tensor of McCrea type, expressed in Schwarzschild coordinates as

$$ds^2 = e^{2(H(r))}dr^2 + r^2d\theta^2 + e^{2(G(r))}d\varphi^2 - e^{2(F(r))}dt^2$$

Can be introduced the pseudo-orthonormal tetradic frame \(\{e_a\}_{a=1,4}\) with the corresponding dual orthonormal base

$$\omega^1 = e^{(H(r))}dr, \quad \omega^2 = rd\theta, \quad \omega^3 = e^{(G(r))}d\varphi, \quad \omega^4 = e^{(F(r))}dt$$

The connection coefficients derived for this metric tensor, are

$$\Gamma^2_{12} = -\Gamma^3_{23} = \frac{1}{r}e^{-\left(H\left(r\right)\right)}$$
$$\Gamma^3_{13} = -\Gamma^4_{34} = G'(r)e^{-\left(H\left(r\right)\right)}$$
$$\Gamma^4_{14} = \Gamma^1_{44} = F'(r)e^{-\left(H\left(r\right)\right)}$$

where we used

$$F'(r) = \frac{dF(r)}{dr} \quad \text{and} \quad G'(r) = \frac{dG(r)}{dr}$$

The Einstein tensor \(G_{ab}\) has the following non-vanishing components

$$G_{11} = \frac{1}{r}\left[G'(r) + F'(r) + rF'(r)G'(r)\right]e^{-2\left(H\left(r\right)\right)}$$
$$G_{22} = \left[G'(r) + G'(r)^2 - H'(r)G'(r) + F''(r) + F'(r)G'(r)\right]e^{-2\left(H\left(r\right)\right)}$$
$$G_{33} = \frac{e^{-2\left(H\left(r\right)\right)}}{r}\left[-H'(r) + rF''(r)^2 - rH'(r)F'(r) + F'(r) + rF'(r)^2\right]$$
$$G_{44} = \frac{e^{-2\left(H\left(r\right)\right)}}{r}\left[-H'(r) + rG''(r) + rG'(r)^2 - rH'(r)G'(r) + G'(r)\right]$$

Considering a charged massive boson, coupled to the electromagnetic field, the \(SO(3,1)\times U(1)\) gauge invariant Lagrangean density has the form [1, 2, 6]:
where we used the definitions

\[ \Phi_a = \Phi - i e A_a \Phi \quad \text{and} \quad \bar{\Phi}_a = \bar{\Phi} + i e A_a \bar{\Phi} \]

(6)

The considered \( \eta^{ab} \) tensor is of minkowskian kind, defined as

\[ \eta^{ab} = \text{diag}[1 \quad 1 \quad 1 \quad -1] \]

(7)

The Maxwell tensor

\[ F_{ab} = A_{b,a} - A_{a,b} \]

is expressed in the terms of the Levi-Civita covariant derivative of the four-potential \( \{A_a\}_{a=1,4} \), i.e.

\[ A_{a,b} = A_{a|b} - A_c \Gamma^c_{ab} \]

(9)

Working in the minimally symmetric ansatz \( A_1 = A_1(r, t) \), \( A_2 = 0 \), \( A_3 = 0 \), \( A_4 = A_4(r, t) \), \( \Phi = \Phi(r, t) \), the single non-vanishing Maxwell tensor component is

\[ F_{41} = -F_{14} = -\left[ \frac{e^{H(r)} A_{1,t} - F'(r) e^{F(r)} A_4 - e^{F(r)} A_{4,r}}{e^{H(r)} e^{F(r)}} \right] \]

(10)

Further it was derived the Klein-Gordon equation, for the scalar spinless field \( \Phi \). The considered evolution equation can be read as

\[ \Box \Phi - m_0^2 \Phi = 2 i e A^c \Phi_{,c} + e^2 A^c A_c \Phi \]

(11)

and it’s hermitic conjugated, with the explicit form

\[ e^{-2(H(r))} \left[ \Phi, \left( \frac{1}{r} - H'(r) + G'(r) + F'(r) \right) + \Phi, \right] - e^{-2(F(r))} \Phi_{,rr} - m_0^2 \Phi = 2 i e \left[ e^{-H(r)} A_t \Phi_x - e^{-F(r)} A_4 \Phi_x \right] + e^2 \Phi \left[ (A_t)^2 - (A_4)^2 \right] \]

(12)

and respectively it’s hermitic conjugated.

The Maxwell system equations [6, 7, 8]:

\[ F_{a,b} = -i e \eta^{ab} \left[ \bar{\Phi} \left( \Phi_b - i e A_b \Phi \right) - \left( \bar{\Phi}_b - i e A_b \bar{\Phi} \right) \Phi \right] \]

(13)

can be written in an explicit form as
\[ e^{-(F(r))}e^{-(H(r))} \left[ -F'(r)A_{4,t} - A_{4,rt} \right] + e^{-2(F(r))}A_{4,t} + \]
\[ = -ie e^{-(H(r))} \left( \overline{\Phi} \Phi_{,r} - \overline{\Phi}_{,r} \Phi \right) - 2e^2 \Phi \Phi A_4 \]
\[ e^{-2(H(r))} \left( F(r) - H'(r)F'(r) + \frac{F'(r)}{r} + G'(r)F'(r) \right) A_4 + \]
\[ + e^{-2(H(r))} \left[ \left( F(r) - H'(r) + \frac{1}{r} + G'(r) \right) A_{4,r} + A_{4,rr} \right] \]
\[ + e^{-H(r)} e^{-F(r)} \left[ \left( -\frac{1}{r} + F'(r) - G'(r) \right) A_{4,t} - A_{4,rt} \right] = \]
\[ = iee^{-F(r)} \left( \overline{\Phi} \Phi_{,r} - \overline{\Phi}_{,r} \Phi \right) + 2e^2 \Phi \Phi A_4 \]

where \( (') \) stands for usual derivative.

In this configuration, the necessary Lorentz condition can be read as
\[ e^{-H(r)} \left[ A_{4,r} + \frac{A_t}{r} + G'(r)A_t + F'(r)A_1 \right] - e^{-F(r)}A_{4,t} = 0 \]

Building up the energy-momentum tensor \([1, 2, 6, 9]\):
\[ T_{ab} = \overline{\Phi}_{,a} \Phi_{,b} + \overline{\Phi}_{,b} \Phi_{,a} + F_{,a} F_{,b} - \eta_{ab} L. \]

it can be derived the Einstein equation
\[ G_{ab} = kT_{ab}, \]
where the tensor \( G_{ab} \) has the explicit form \((4)\).

The Einstein equations can be read as
\[ \frac{1}{r} \left[ G'(r) + F'(r) + rF'(r)G'(r) \right] e^{-2(H(r))} = \]
\[ = \kappa \left[ \left( \overline{\Phi_j} \Phi_{,j} + \overline{\Phi}_{,j} \Phi_j \right) - m_0^2 \overline{\Phi} \Phi - \frac{1}{2} (F_{14})^2 \right] \]
\[ \left[ G'(r) + G'(r)^2 - H'(r)G'(r) + F''(r) + \right. \]
\[ + F'(r)^2 - H'(r)F'(r) + F'(r)G'(r) \left. \right] e^{-2(H(r))} = \]
\[ = -\kappa \left[ \left( \overline{\Phi_j} \Phi_{,j} - \overline{\Phi}_{,j} \Phi_j \right) + m_0^2 \overline{\Phi} \Phi - \frac{1}{2} (F_{14})^2 \right] \]
respectively
\[ \frac{1}{r} \left[ -H'(r) + rG'(r) + rG'(r)^2 - rH'(r)G'(r) + G'(r) \right] e^{-2(H(r))} = \]
\[ = \kappa \left[ \left( \overline{\Phi_j} \Phi_{,j} + \overline{\Phi}_{,j} \Phi_j \right) + m_0^2 \overline{\Phi} \Phi + \frac{1}{2} (F_{14})^2 \right] \]
where the energy-momentum tensor $T_{ab}$ has the explicit form

$$T_{11} = -m_0^2 \Phi \Phi + e^2 \Phi \Phi \left( A_1^2 + A_4^2 \right) +$$

$$+ i e \left[ \frac{A_1}{e^{(H(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) + \frac{A_4}{e^{(F(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) \right] +$$

$$+ \frac{1}{e^{(2H(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) + \frac{1}{e^{(2F(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) +$$

$$- \frac{1}{2} e^{H(r)} \frac{1}{e^{2F(r)}} \left[ A_{1,\mu} - F'(r) e^{F(r)} A_{4,\mu} - e^{F(r)} A_{4,\mu} \right] ^2$$

$$T_{22} = -m_0^2 \Phi \Phi + e^2 \Phi \Phi \left( A_1^2 - A_4^2 \right) +$$

$$- \left[ \frac{A_1}{e^{(H(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) - \frac{A_4}{e^{(F(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) \right] +$$

$$- \frac{1}{e^{(2H(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) - \frac{1}{e^{(2F(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) +$$

$$- \frac{1}{2} e^{H(r)} \frac{1}{e^{2F(r)}} \left[ A_{1,\mu} - F'(r) e^{F(r)} A_{4,\mu} - e^{F(r)} A_{4,\mu} \right] ^2$$

$$T_{33} = T_{22}$$

$$T_{44} = m_0^2 \Phi \Phi + e^2 \Phi \Phi \left( A_1^2 + A_4^2 \right) +$$

$$+ \left[ \frac{A_1}{e^{(H(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) - \frac{A_4}{e^{(F(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) \right] +$$

$$+ \frac{1}{e^{(2H(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) + \frac{1}{e^{(2F(r))}} \left( \Phi_{\Phi, \tau} - \Phi_{\Phi, \tau} \right) +$$

$$+ \frac{1}{2} e^{H(r)} \frac{1}{e^{2F(r)}} \left[ A_{1,\mu} - F'(r) e^{F(r)} A_{4,\mu} - e^{F(r)} A_{4,\mu} \right] ^2$$

For solving we start with the physical reasonable assumptions that the charged scalar field is the main source of both the electromagnetic and gravitational fields. Neglecting the feedback of gravity, in the first order approximation, could write the equation for potential $\Phi$. For a flat space–time structure, should be imposed $H(r) = 0$, $G(r) = \ln(r) + \mu$ and $F(r) = 0$. In this anzatz, the Klein–Gordon equation can be read as:

$$\Phi_{\mu, \nu} - \frac{2}{r} \Phi_{\mu, \nu} - \Phi_{\mu, \nu} - m_0^2 \Phi = 0$$

and it’s hermitic conjugated, with the spherically kind solutions
\[
\Phi = \frac{N}{r} e^{i(\omega t - kr)}, \quad \bar{\Phi} = \frac{N}{r} e^{-i(\omega t - kr)}
\]  

(24)

Using the same conditions for a flat space–time, from Maxwell’s equations (14–15), can be read:

\[
A_{1,rr} + \frac{1}{2r} A_{1,r} - \frac{1}{2r^2} A_1 - A_{1,tt} = -2ek \frac{|N|^2}{r^2}
\]  

(25)

and respectively

\[
A_{4,rr} + \frac{2}{r} A_{4,r} - A_{4,tt} = 2e\omega \frac{|N|^2}{r^2}
\]  

(26)

From (25) can be considered a particular solution [9, 10, 11] of the form

\[
A_1 = ek |N|^2
\]  

(27)

while, from the second Maxwell’s equation (26), can be found that

\[
A_4(r, t) = 2\omega k |N|^2 \log\left(\frac{r}{r_0}\right) + 2ek \frac{|N|^2}{r^2} t
\]  

(28)

In order to achieve a first order evaluation for the fields’ system magnitudes, should be followed the anterior papers program [1, 2, 6, 11]. Introducing these first order perturbative solutions (27–28) in the Einstein’s system equations (19–21), can be read [12]

\[
\frac{1}{r^2} \left[ G''(r) + F'(r) + rF'(r)G'(r) \right] =
\]

\[
= \kappa \left[ \frac{N^2}{r^4} - 2e^2 m_0^2 \frac{N^4}{r^2} + 4e^2 m_0^2 \frac{N^4}{r^2} \log\left(\frac{r}{r_0}\right) \right]
\]  

(29)

and

\[
G''(r) + G'(r)^2 - H'(r)G'(r) + F''(r) + F'(r)^2 - H'(r)F'(r) + F'(r)G'(r) =
\]

\[
= -k \left[ \frac{|N|^2}{r^4} + 2e^2 m_0^2 \frac{|N|^2}{r^2} + 4e^2 m_0^2 \frac{|N|^4}{r^2} \log\left(\frac{r}{r_0}\right) \right]
\]  

(30)

respectively

\[
\frac{1}{r^2} \left[ -H'(r) + rG''(r) + rG'(r)^2 - rH'(r)G'(r) + G'(r) \right] =
\]

\[
= k \left[ \frac{|N|^2}{r^4} + 2m_0^2 \frac{|N|^2}{r^2} + 2e^2 m_0^2 \frac{|N|^4}{r^2} + 4e^2 m_0^2 \frac{|N|^4}{r^2} \log\left(\frac{r}{r_0}\right) \right]
\]  

(31)
For this non-linear differential equations system, in an analytical approach, could be considered a first order approximate solution [11, 13, 14]. Particular first order solutions for the metric tensor functions can be read as:

\[ H(r) = -2ke^2m_0^2|N|^4 \log \left( \frac{r}{r_0} \right) + \kappa \frac{1}{2} \frac{|N|^2}{r^2} + 
\]
\[ +2ke^2m_0^2|N|^4 \log (r) + 2 \log \left( \log \left( \frac{r}{r_0} \right) \right)^2 - \frac{1}{2} + \frac{1}{4r^2e^2m_0^2|N|^2} \approx \]
\[ \approx c(1 - \log(r_0))r^2 - 4c(1 - 2\log(r_0)) + \frac{b}{2r^2} - 
\]
\[ -2c\log(r_0)(\log(r_0) - 3) - (\log(r) - 2c) \]

and,

\[ G(r) = \frac{1}{2r^2} \kappa |N|^2 \left[ -4e^2 |N|^2m_0^2r^2 \log \left( \frac{r}{r_0} \right) - 1 + 4e^2 |N|^2m_0^2r^2 \log \left( \frac{r}{r_0} \right)^2 \right] \approx \]
\[ \approx \frac{b}{2r^2} + 2rc(1 - 2\log(r_0)) + C_1 \]

where we used the definitions

\[ a = \kappa m_0^2 |N|^2 \quad b = \kappa |N|^2 \quad c = \kappa e^2 m_0^2 |N|^4 \]

For the third function, was found

\[ F(r) = -\frac{1}{2r \log(r)} + C_2 + C_3r - \log \left( \frac{r}{r_0} \right) \]

For a complete view, let’s end this discussion with the system’s electric charge evaluation [11, 14, 15, 16]

\[ Q = \int j_4 dV \]  

where the fourth component of the electric charge current is

\[ j_4 = e^{\langle -H(r) \rangle}ie \left[ i\Phi \mathcal{P}_{ij} - \Phi \mathcal{P}_{ij} \right] + 2ie \langle H(r) \rangle \Phi \mathcal{P}_{ij} \]

Using the found particular solutions of the Klein–Gordon–Maxwell–Einstein system equations (24), (27) and (32), the electric charge magnitude is of the form
The finite obtained value of the system’s electric charge is an interesting result regarding the previous evaluations and discussions [1, 2, 11, 14].

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