HWAKING RADIATION AS TUNNELING FOR KERR-NEWMAN-de SITTER BLACK HOLE

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When a black hole radiates particles, it loses energy and shrinks, the horizon contracts from its original radius to a new smaller radius. This leads to the separation between the initial and final radius, which sets the barrier for the particles to tunnel. We develop the work of Parikh to the axis symmetry rotating Kerr-Newman-de Sitter spaces, and the Parikh method to calculate the rate of the Hawking radiation. Black hole horizon and cosmological horizon have the uniform expression \( r = e^\Delta \). It is also proven that the energy spectrum deviates from exact thermality. So, we realize that cosmological horizon Hawking radiation can indeed be viewed as particle tunneling process. The coordinate conversion, given on the base of investigated object un-axes symmetric and un-thermo equilibrium, should be applied to all kinds of complex stationary spacetime. The conclusion obtained is universal.

Key words: Kerr-Newman-de Sitter black hole; Hawking radiation; Quantum theory.

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1 INTRODUCTION

In the 1970s, Hawking declared his astonishing discovery that radiation from the black hole can be completed by tunneling. When a virtual particle pair is created by vacuum fluctuations just near the horizon, the negative energy particles is absorbed by the black hole, while the positive energy virtual particle appears to be the Hawking radiation. Gibbons and Hawking also demonstrated that the energy spectrum of radiation is exactly thermal [2].

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Recently much attention has focused on studying the de Sitter (dS) space and asymptotically dS space. It is motivated at least by the following two aspects. First, recent analysis of astronomical data for supernova indicates that there is a positive cosmological constant in our universe [3–5]. Thus our universe might approach to a dS phase in the far future [6, 7]. Second, defined in a manner analogous to the AdS/CFT correspondence, an interesting proposal, the so-called dS/CFT correspondence, has been suggested recently that there is a dual between quantum gravity on a dS space and an Euclidean conformal field (CFT) on a boundary of the dS space [8]. Therefore, the research of dS spacetime thermodynamics is one of the most interesting topic attracted more and more theoretical Physicians. The recent calculations of Parikh and Wilczek indicate that Hawking radiation of a black hole is not pure thermal if the self-gravitation is considered [9–12]. They treat Hawking radiation as a tunneling process, and their key insight is to find a coordinate system well behaved at the event horizon to calculate the emission rate. They have calculated the emission spectrum of the Schwarzschild black hole. Their result is consistent with an underlying unitary theory. Recently, the research groups in Ref. [13, 14] applied this method to axes symmetry space, and obtained significance conclusion. In this letter, we extend the existed research to the universal Kerr-Newman-de Sitter spaces, and give out a universal coordinate conversion. The time direction of this new coordinate is a Killing vector and the metric is smooth at the horizon. The superiority of this coordinate can be presented by calculating the tunneling rate from black hole horizon and cosmological horizon. For Kerr-Newman-de Sitter spaces, we obtain the conclusion that the tunneling rate from black hole horizon and cosmological horizon is the exponent of the different in Bekenstein-Hawking entropy, $\Delta S$, before and after emission.

2. TUNNELING PROCESS

The line element of the Kerr-Newman-de Sitter (KNdS) black hole can be written in the form \cite{15}

$$
\begin{align*}
    ds^2 &= -A(r, \theta)dt^2 + \frac{dr^2}{B(r, \theta)} + \frac{\rho^2}{\Delta_0} d\theta^2 + \frac{\Sigma^2 \sin^2 \theta}{\rho^2 \Xi^2} (d\phi - \Omega(r, \theta)dt)^2,
\end{align*}
$$

(1)

where

$$
\begin{align*}
    A(r, \theta) &= \frac{\rho^2 \Delta_r \Delta_0}{\Sigma^2}, & B(r, \theta) &= \frac{\Delta_r}{\rho^2}, & \rho^2 &= r^2 + a^2 \cos^2 \theta, & \Xi = 1 + \frac{a^2}{l^2}, \\[4pt]
    \Delta_r &= (r^2 + a^2) \left(1 - \frac{r^2}{l^2}\right) - 2Mr + Q^2, & \Delta_0 &= 1 + \frac{a^2}{l^2} \cos^2 \theta, \\[4pt]
    \Sigma^2 &= \Delta_0 (r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta, & \Omega(r, \theta) &= \frac{a[\Delta_0 (r^2 + a^2) - \Delta_r] \Xi}{\Xi^2}.
\end{align*}
$$
The limiting value of $\Omega(r, \theta)$ as $r \rightarrow r_H$ is the angular velocity of the horizon

$$\Omega(r_H) = \frac{a^2}{r_H^2 + a^2}. \quad (2)$$

with line element (1), spacetime axes un-perpendicular, without simultaneity surface is given, we use dragging coordinate

$$\frac{d\phi}{dt} = -\frac{g_{03}}{g_{33}} = \Omega(r, \theta). \quad (3)$$

The line element of the Kerr-Newman-de Sitter Black Hole can be rewritten as

$$dS^2 = -A(r, \theta)dt^2 + \frac{dr^2}{B(r, \theta)} + \frac{r^2}{\Delta_0}d\theta^2. \quad (4)$$

The largest root of the quartic algebraic equation defines the horizon

$$\Delta_r = 0, \quad (r_H^2 + a^2) \left(1 - \frac{r_H^2}{r^2}\right) - 2Mr_H + Q^2 = 0. \quad (5)$$

It is easy to find that this metric is singularity at the position $r = r_H$. To describe tunneling, we need a coordinate system that is regular at the horizon. Let

$$T = t + \int dr \sqrt{\frac{1 - g(r)}{A(r, \theta)B(r, \theta)}}. \quad (6)$$

where $g(r) = 1 - \frac{r_H}{r}$.

Thus the line element described in Eq. (4) reads

$$dS^2 = -A(r, \theta)dt^2 + 2A(r, \theta) \sqrt{\frac{1 - g(r)}{A(r, \theta)B(r, \theta)}} + \frac{g(r)dr^2}{B(r, \theta)} + \frac{r^2}{\Delta_0}d\theta^2. \quad (7)$$

With the new metric we can now solve the only curves that are both radial and null.

$$0 = \left(\frac{dS}{dT}\right)^2, \quad 0 = -A(r, \theta) + 2\sqrt{\frac{[1 - g(r)]A(r, \theta)}{B(r, \theta)}} \dot{r} + \frac{\dot{g}(r)}{B(r, \theta)} \dot{r}^2. \quad (8)$$

$$\dot{r} = \frac{1}{\sqrt{G(r, \theta)}} \left[\pm 1 - \sqrt{1 - g(r)}\right], \quad G(r, \theta) = \frac{g^2(r)}{A(r, \theta)B(r, \theta)}.$$

For the black hole, the Abbott and Deser (AD) [16] mass is conservation while the mass of the black hole decreases. Since black hole horizon is sphere and the
metric is of spherical symmetry. It’s reasonable to regard the outgoing particle as an $s$-wave, i.e. a shell of energy. When a shell of energy $\omega$ moves in the geodesic of the spacetime, the mass of the black hole becomes $E - \omega$, so Eq. (7) should be modified by replacing $r_H(E)$ with $r_H(E - \omega)$. Assume that the outgoing wave is traced back toward the horizon. Its wavelength, as measured by local fiducial observers, will be blue-shifted. Near the horizon, the radial wave number approaches infinity, so that the geometrical optics limit or the WKB approximation is appropriate [12–14].

The $s$-wave function of the outgoing positive energy particle can be expressed as $\psi(r) = e^{iHr}$, so the rate of tunneling $\Gamma$ takes the form

$$\Gamma \sim \exp[-2ImI].$$

With

$$I = \int_{r_i}^{r_f} p_r dr = \int_{r_i}^{r_f} dp_r dr.$$

where $p_r$ is the radial momentum, and the initial radius $r_i$ corresponds to the pair-creation site, which should be slightly inside the initial horizon. The final radius $r_f$ is slightly outside the final position of the horizon and is actually less than $r_i$.

First the classical momentum is expanded into an integral. Next Hamilton’s equation is used to transform variables from momentum to energy.

$$\frac{dH}{dp} = \dot{r}$$

$$I = \int_{r_i}^{r_f} p_r dr = \int_{r_i}^{r_f} dH = \int_{r_i}^{r_f} \frac{dH}{\dot{r}} dr = \int_{r_i}^{r_f} \int_{E_i}^{E} \frac{dH}{\dot{r}} dr,$$

In this last equation we have used the fact that if the particles have tunneled out, then the black hole will have lost some energy $\omega$.

Next we switch integration variables from $H$ to the particle energy $\omega$ [14].

$$H = E - \Omega J, \quad dH = (1 - a\Omega(r, \theta)/\Xi)dE,$$

$$Im I = Im \int_{r_i}^{r_f} dE \int_{E_i}^{E} \frac{(1 - a\Omega(r, \theta)/\Xi)dE}{\dot{r}}.$$

The imaginary part of the action for the outgoing, positive energy particle is
Hawking radiation as tunneling

Substituting \( u = r^{1/2} \), it is easy to find that there is a pole at \( u = r_H^{1/2} \), we have

\[
\text{Im} \ Int = \text{Im} \left( \int_{r_i}^{r_f} \sqrt{G(r, \theta)(1 - a \Omega(r, \theta)/\Xi)} \frac{dE}{1 - \sqrt{r_H/r}} \right)
\]  

(15)

Substituting \( u = r^{1/2} \), it is easy to find that there is a pole at \( u = r_H^{1/2} \), we have

\[
\text{Im} \ Int = 2 \text{Im} \left( \int_{E_i}^{E_f} dE \int_{r_i^{1/2}}^{r_f^{1/2}} \frac{\sqrt{G(u, \theta)(1 - a \Omega(r, \theta)/\Xi)} u^2}{u - \sqrt{r_H}} du \right)
\]

\[
= 2 \text{Im} \lim_{\epsilon \to 0} \left( \int_{E_i}^{E_f} dE \int_{r_i^{1/2}}^{r_f^{1/2}} \frac{\sqrt{G(u, \theta)(1 - a \Omega(r, \theta)/\Xi)} u^2}{u - r_H^{1/2} + i\epsilon} du \right)
\]

(16)

where \( G(u, \theta) = \frac{g^2(u)}{A(u, \theta)B(u, \theta)} \). In Eq. (16), in order to integrate across the singularity at point \( r = r_H \), we substitute \((u - r_H^{1/2} + i\epsilon)\) for \((u - r_H^{1/2})\). This method, however, is not strict in mathematics. In recent years, this problem was dealt with by considering the quantum correction of the horizon \[17\].

According to formulation (10), we know that the real part of \( I \) only contributes a phase to the tunneling rate while the imaginary part of \( I \) contributes to the rate amplitude. What we are interested in is the imaginary part of the integral then

\[
\text{Im} \ Int = -2\pi \int_{E_i}^{E_f} dE \sqrt{f(r_H)(1 - a \Omega(r_H)/\Xi)} r_H.
\]

(17)

when \( u \to r_H^{1/2}, \ G(u, \theta) \) is the form of \( 0/0 \), thus we use L’Hospital principle to obtain

\[
\lim_{u \to r_H^{1/2}} G(u, \theta) = \frac{1}{r_H^2 A'(r_H)B'(r_H)}.
\]

Therefore, \( f(r_H) \) in equation (17) is \( f(r_H) = \frac{1}{r_H^2 A'(r_H)B'(r_H)} \).

The AD mass \( E \) and angular momentum \( J \) associated with the black hole horizon are

\[
E = \frac{1}{2 \Xi r_H^2} \left[(r_H^2 + a^2) \left(1 - \frac{r_H^2}{l^2}\right) + Q^2\right], \quad J = \frac{a}{2 \Xi r_H^2} \left[(r_H^2 + a^2) \left(1 - \frac{r_H^2}{l^2}\right) + Q^2\right],
\]

we obtain \( dE = \chi(r_H)dr_H \), with
\[
\chi(r_H) = \frac{1}{2\Xi} \left[ 1 - \frac{a^2}{l^2} - \frac{r_a^2}{r_H^2} - 3 \frac{r_m^2}{r_H^2} - \frac{Q^2}{r_H^2} \right],
\]
equation (17) can be rewritten as
\[
\text{Im} l = \int_{r_f}^{r_i} dr_H F(r_H).
\tag{18}
\]
with \( F(r_H) = -2\pi r_H \chi(r_H) \sqrt{f(r_H)} \left[ 1 - aQ(r_H)/\Xi \right] \)

We Taylor expand \( F(r_H) \) in \( r_i \), we have
\[
F(r_H) = F(r_i) + F'(r_i)(r_H - r_i) + \frac{F''(r_i)}{2}(r_H - r_i)^2 + \ldots
\tag{19}
\]
so from formula (18), we have
\[
\text{Im} l = F(r_i) \Delta r_H + \frac{1}{2} F'(r_i)(\Delta r_H)^2 + \frac{F''(r_i)}{3!}(\Delta r_H)^3 + \ldots
\tag{20}
\]
where \( \Delta r_H = r_f - r_i \) is the variance of the radius of horizon, which is actually negative. The tunneling rate can be expressed as
\[
\Gamma = \exp \left[ -2\text{Im} l \right] = \exp \left\{ -2 \left[ F(r_i) \Delta r_H + \frac{1}{2} F'(r_i)(\Delta r_H)^2 + \frac{F''(r_i)}{3!}(\Delta r_H)^3 + \ldots \right] \right\}. \tag{21}
\]

Kerr-Newman-de Sitter the black hole entropy is \([15, 18]\)
\[
S_{BH} = \frac{\pi}{2} (r_H^2 + a^2).
\tag{22}
\]

We could obtain
\[
\frac{dS_{BH}}{dr_H} = -2 F(r_H).
\tag{23}
\]
Thus the variance of entropy before and after emission is
\[
\Delta S_{BH} = S_{BH}(r_f) - S_{BH}(r_i) = \frac{dS_{BH}}{dr_H} \Delta r_H + \frac{1}{2!} \frac{d^2 S_{BH}}{dr_H^2} (\Delta r_H)^2 + \frac{1}{3!} \frac{d^3 S_{BH}}{dr_H^3} (\Delta r_H)^3 + \ldots
\tag{24}
\]
\[
= - \left[ F(r_i) \Delta r_H + \frac{1}{2!} F'(r_i)(\Delta r_H)^2 + \frac{F''(r_i)}{3!}(\Delta r_H)^3 + \ldots \right].
\]
Comparing each term of Eq. (24) with Eq. (21), we find that the rate of emission agrees with [9, 10]
\[ \Gamma \sim e^{-2m \ln l} = \exp[\Delta S_{BH}] = e^{-\beta(\omega-\omega_0) + \alpha_1(\omega-\omega_0)^2 + \alpha_2(\omega-\omega_0)^3 + \ldots}. \] (25)

If we only take the first term but neglect the higher-order terms in the expression (25), the energy spacetime takes the form 
\[ e^{-\beta(\omega-\omega_0) + \alpha_1(\omega-\omega_0)^2} \] with the precisely inverse of temperature
\[ \frac{1}{\beta} = T_H = \frac{r_H}{4\pi(r_H^2 + a^2)} \left[ \left( 1 - \frac{a^2}{l^2} \right) - \frac{a^2}{r_H^2} - 3 \frac{r_H^2}{r_H^2} \frac{Q^2}{r_H^2} \right]. \] (26)

where \( \omega_0 = \omega \alpha \Omega(r_H) \). So we confirm that the Hawking radiation can be viewed as particle tunneling.

The cosmological horizon has associated thermodynamic quantities
\[ \tilde{S} = \frac{\pi}{2} \left( r_c^2 + a^2 \right), \quad \tilde{E} = -\frac{1}{2\pi r_c} \left( r_c^2 + a^2 \right) \left( 1 - \frac{r_c^2}{l^2} \right) + Q^2, \]
\[ \tilde{J} = \frac{a}{2\pi r_c^2} \left( r_c^2 + a^2 \right) \left( 1 - \frac{r_c^2}{l^2} \right) + Q^2, \quad \Omega(r_c) = -\frac{a\Xi}{r_c^2 + a^2}. \] (27)

making use of the above calculate method with \( g(r) = 1 - \frac{r_c}{r} \). We could obtain
\[ \Gamma = e^{-\beta_c(\omega-\omega_0) + \alpha'_1(\omega-\omega_0)^2 + \alpha'_2(\omega-\omega_0)^3 + \ldots}, \] (28)

If we only take the first term but neglect the higher-order terms in the expression (28), the energy spectrum takes the form 
\[ e^{-\beta_c(\omega-\omega_0)} \] with the precisely inverse of temperature
\[ \frac{1}{\beta_c} = T_c = \frac{r_c}{4\pi(r_c^2 + a^2)} \left[ \left( 1 - \frac{a^2}{l^2} \right) + \frac{a^2}{r_c^2} + 3 \frac{r_c^2}{l^2} + Q^2 \right], \] (29)

where \( \omega_0 = \omega \alpha \Omega(r_c) \). So, in KNDS black hole, we realize that cosmological horizon Hawking radiation can indeed be viewed as particle tunneling process.

3. DISCUSSION

From the above discussion, we realize that either the black hole event horizon or cosmological horizon Hawking radiation can indeed be viewed as particle tunneling process. However, if we take into account the energy conservation, which has a correction and the higher-order terms then the spectrum, is not precisely thermal.
During the Hawking radiation, energy conservation plays an important role. The black hole loses the energy of particle and then shrinks. It is the contraction of the horizon that supplies the barrier through which the particle tunnels. Energy conservation also causes the energy spectrum of radiation to deviate from exact thermality. Probably, there is other information besides temperature we can obtain from the spectrum. To calculate the rate of radiation, we expand the complicated integration as Taylor series and compare each term with the variance of entropy instead of integrating directly.

Our method should be suitable to calculate any stationary spacetime. The given coordinate conversion (6) is universal according to the universality of Kerr-Newman-de Sitter spaces.

REFERENCES