UNSTEADY MHD FLOW WITH HEAT TRANSFER IN A DIVERGING CHANNEL

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In this paper, we investigate the combined effects of unsteadiness and a transversely imposed magnetic field on viscous incompressible fluid flow and heat transfer in a slowly varying exponentially diverging symmetrical channel. The nonlinear governing equations are obtained and solved analytically using a perturbation technique. Graphical results are presented and discussed quantitatively for various flow characteristics like axial velocity, temperature, axial pressure gradient, wall shear stress and Nusselt number.

Key words: Flow unsteadiness, diverging channel, magnetic field, conducting fluid.

1. INTRODUCTION

MHD flow in diverging channels/ducts has important applications in MHD pumps and generators, liquid metal magnetohydrodynamics and physiological fluid flow. In liquid metal magnetohydrodynamics, magnetic fields are used to levitate samples of liquid metal, to control their shape and to induce internal stirring for the purpose of homogenisation of the final product [6]. Magnetic fields are also used to control the natural convection of semiconductor melts such as silicon or gallium arsenide to improve crystal quality [7]. In physiological fluid flow, Barnothy has reported experiments [4] where the heart rate decreased by exposing biological systems to an external magnetic field. In this line of application is magnetic resonance imaging (MRI), a technique for obtaining high resolution images of various organs within the human body in the presence of a magnetic field.

In 1937, Hartmann and Lazarus [11] studied the influence of a transverse uniform magnetic field on the flow of a viscous incompressible electrically conducting fluid between two infinite parallel stationary and insulating plates. Since then, this pioneering work in MHD flow has received much attention and has been extended in numerous ways. Closed form solutions for the velocity fields under different physical effects have been studied in [8] and [16]. A heat

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transfer problem has been studied in [16] and [2]. Hall effects, in the case when the magnetic field is strong, have been incorporated in [16]. The effect of variable properties (temperature dependent viscosity and thermal conductivity) has been studied in [3]. An analysis of steady incompressible Hartmann flow through ducts of varying cross-section has been done in [12]; Pure oscillatory MHD flow of blood through channels of varying cross-section has been studied in [15];

In the present work, unsteady Hartmann flow is studied in a diverging channel of slowly varying cross section. The unsteadiness is caused by an oscillating flux with constant pulse \( m \). The concept of a slowly varying flow forms the basis of a large class of fluid flow problems [6]. It is therefore interesting to study the effect of a magnetic field to the flow in this diverging geometry.

2. MATHEMATICAL FORMULATION

We consider the effect of a uniform transverse magnetic field \( \mathbf{B} \) on unsteady two dimensional electric conducting fluid flow whose velocities are given by

\[
\mathbf{q} = u(x, y, t) \mathbf{i} + v(x, y, t) \mathbf{j}
\]

through a symmetric diverging channel \( \{D: -\infty < x < +\infty, -b(x) < y < b(x)\} \) where \((x, y)\) are Cartesian co-ordinates such that \(0x\) is the axis of symmetry of the channel and \( y = \pm b(x) \) are the rigid and impermeable walls of the channel. The walls of the channel are kept at a constant temperature \( T_w \). We assume that the fluid is incompressible with uniform properties \( i.e. \) density \( \rho \), kinematic viscosity \( \nu \) and electrical conductivity \( \sigma \).

A volume flux with oscillating frequency \( \delta \) and pulse \( m \) is prescribed as

![Fig. 1. – Problem geometry.](image)
A uniform magnetic force is applied in the y-direction. A very small magnetic Reynolds number is assumed and therefore the induced magnetic field is neglected. Two key physical effects occur when the fluid moves into the magnetic field. The first one is that an electric field $E$ is induced in the flow. We will assume that there is no excess charge density and therefore that $\nabla \cdot E = 0$. Neglecting the induced magnetic field implies that $\nabla \times E = 0$ and therefore the induced electric field is negligible. The second key effect is dynamical i.e. a Lorentz force $(J \times B)$, where $J$ is the current density acts on the fluid and modifies its motion. Therefore there is a transfer of energy $(J \cdot E)$ from the electromagnetic field to the fluid. In this study, relativistic effects are neglected and $J$ is given by Ohm’s law:

$$J = \sigma q \times B.$$  

Taking into account the Lorentz force and the energy transfer, the equations governing the unsteady flow of a Newtonian fluid are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \nabla^2 u - \frac{\sigma B_0^2}{\rho} u,$$  

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \nabla^2 v,$$  

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} =$$

$$= \frac{k}{c_p} \nabla^2 T + \frac{\nu}{\rho c_p} \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right) + \frac{\sigma B_0^2}{\rho c_p} u^2,$$

with conditions:

Symmetry : $\frac{\partial u}{\partial y} = 0, \ v = 0, \ \frac{\partial T}{\partial y} = 0$ on $y = 0.$  

Non – slip : $u + v \frac{db}{dx} = 0, \ T = T_w$ on $y = b(x)$.

It is convenient to introduce the stream function $\Psi$ defined by $u = \partial \Psi / \partial y, \ v = -\partial \Psi / \partial x$ and after eliminating the pressure term from equations (5) and (6), we obtain
\[-\nabla^2 \Psi = \omega, \quad (10)\]

\[\frac{\partial \omega}{\partial t} + \frac{\partial (\Psi, \omega)}{\partial (y, x)} = \nu \nabla^2 \omega + \frac{\sigma B^2}{\rho} \frac{\partial^2 \Psi}{\partial y^2}, \quad (11)\]

\[\frac{\partial T}{\partial t} + \frac{\partial (T, \Psi)}{\partial (y, x)} = \frac{k}{pc_p} \nabla^2 T + \frac{\nu}{c_p} \left( \frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x^2} \right)^2 + \frac{4\nu}{c_p} \left( \frac{\partial^2 \Psi}{\partial x \partial y} \right)^2 + \frac{\sigma B^2}{pc_p} \left( \frac{\partial \Psi}{\partial y} \right)^2. \quad (12)\]

The corresponding boundary conditions are

\[\frac{\partial^2 \Psi}{\partial y^2} = 0, \quad \Psi = 0, \quad \frac{\partial T}{\partial y} = 0 \quad \text{on} \ y = 0 \quad (13)\]

\[\frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial b} = 0, \quad T = T_o, \quad \Psi = Q(1 + me^{i\delta t}) \quad \text{on} \ y = b(x). \quad (14)\]

The function \(b(x)\) is assumed to depend upon a small parameter \(\varepsilon\) such that

\[b(x, \varepsilon) = a_0 S \left( \frac{ex}{a_0} \right) \quad (0 < \varepsilon = \frac{a_0}{L} \ll 1), \quad (15)\]

where \(a_0\) is the constant characteristic half width of the channel, \(L\) is the constant characteristic length of the channel and \(S\) is the function describing the channel wall divergence geometry. This assumption helps us to simplify the problem by writing the equations in non-dimensional form. To achieve this, we define \(T_0\) the reference temperature and the following non-dimensional quantities:

\[\omega' = \frac{a_0 \omega}{Q}, \quad x' = \frac{ex}{a_0}, \quad y' = \frac{y}{a_0}, \quad \Psi' = \frac{\Psi}{Q}, \quad t' = \frac{T - T_0}{T_o - T_0}, \quad \alpha = \frac{\delta a_0}{\nu}, \quad (16)\]

\[Re = \frac{Qc}{\nu}, \quad P' = \frac{ea_0^2}{\rho \nu Q} P.\]

In terms of these non-dimensional quantities and after neglecting terms of order \(\varepsilon^2\) and higher order as well as the primes symbol for clarity, we obtain

\[-\frac{\partial^2 \Psi}{\partial y^2} = \omega \quad (17)\]

\[\frac{\partial^2 \omega}{\partial y^2} - \alpha \frac{\partial \omega}{\partial t} = Re \left( \frac{\partial^2 (\Psi, \omega)}{\partial (y, x)} - \frac{\dot{\Omega} \partial \Psi}{\partial y^2} \right). \quad (18)\]

\[\frac{\partial^2 \Theta}{\partial y^2} - \alpha P' \frac{\partial \Theta}{\partial t} = Re P' \left( \frac{\partial^2 (\Theta, \Psi)}{\partial (y, x)} - \Omega E \left( \frac{\partial^2 \Psi}{\partial y^2} \right)^2 \right) - E, P' \left( \frac{\partial^2 \Psi}{\partial y^2} \right)^2. \quad (19)\]
where \( E_c = Q^2/a_0^2(T_\omega - T_0)c_p \) is the Eckert number, \( P_r = \rho c_p v/k \) the Prandtl number, \( \alpha \) the Womersley number, \( Re \) the effective flow Reynolds number and \( \Omega = \sigma B_0^2 a_0 L/\rho Q \) is the magnetic field intensity parameter. The boundary conditions are

\[
\frac{\partial^2 \Psi}{\partial y^2} = 0, \quad \Psi = 0, \quad \frac{\partial \theta}{\partial y} = 0 \quad \text{on} \quad y = 0
\]

\[
\frac{\partial \Psi}{\partial y} = 0, \quad \theta = 1, \quad \Psi = 1 + me^{it} \quad \text{on} \quad y = S(x).
\]

The problem is now completely specified by the system of equations (17)–(21).

### 3. Method of Solution

Due to the nonlinear nature of the equations (17)–(19), it is convenient to take a power series expansion in the effective flow Reynolds number \( Re \) as follows:

\[
\Psi = \sum_{j=0}^{\infty} Re^j (\Psi_j + me^{it} \Psi_j), \quad \omega = \sum_{j=0}^{\infty} Re^j (\omega_j + me^{it} \omega_j),
\]

\[
\theta = \sum_{j=0}^{\infty} Re^j (\theta_j + me^{it} \theta_j),
\]

where \( \Psi_j, \ \Psi_j, \ \omega_j, \ \omega_j, \ \theta_j \) and \( \theta_j \) are functions of \( S(x) \) and \( y \). It is important to note that the real part of the equation (22) forms the solution of the problem which is physically meaningful. Substituting equation (22) into equations (17)–(19) and collecting terms of like order of \( Re \) and \( me^{it} \), we obtain Zeroth order:

\[
\frac{\partial^2 \omega_0}{\partial y^2} = \lambda_1^2 \omega_0
\]

\[
\frac{\partial^2 \Psi_0}{\partial y^2} = -\omega_0
\]

\[
\frac{\partial^2 \theta_0}{\partial y^2} - \lambda_2^2 \theta_0 = -2E_c P_r \frac{\partial^2 \Psi_0}{\partial y^2} \frac{\partial^2 \Psi_0}{\partial y^2}
\]

\[
\frac{\partial^2 \omega_{0s}}{\partial y^2} = 0
\]
\[
\frac{\partial^2 \Psi_{0s}}{\partial y^2} = -\omega_{0s} \tag{27}
\]

\[
\frac{\partial^2 \theta_{0s}}{\partial y^2} = E_r P_r \left( \frac{\partial \Psi_{0s}}{\partial y} \right)^2 \tag{28}
\]

where \( \lambda_2 = \alpha_i \) and \( \lambda_3 = P_r \alpha_i \) and we require that \( P_r \neq 1 \). The boundary conditions are

\[
\frac{\partial^2 \Psi_0}{\partial y^2} = \frac{\partial^2 \Psi_{0s}}{\partial y^2} = 0, \quad \Psi_0 = \Psi_{0s} = 0, \quad \frac{\partial \theta_0}{\partial y} = \frac{\partial \theta_{0s}}{\partial y} = 0 \quad \text{on} \quad y = 0. \tag{29}
\]

\[
\frac{\partial \Psi_0}{\partial y} = \frac{\partial \Psi_{0s}}{\partial y} = 0, \quad \Psi_0 = \Psi_{0s} = 1, \quad \theta_0 = 0, \quad \theta_{0s} = 1 \quad \text{on} \quad y = S(x). \tag{30}
\]

First order:

\[
\frac{\partial^2 \omega_1}{\partial y^2} - \lambda_1^2 \omega_1 = \frac{\partial (\Psi_0, \omega_{0s})}{\partial (y, x)} + \frac{\partial (\Psi_{0s}, \omega_0)}{\partial (y, x)} - \Omega \frac{\partial^2 \Psi_0}{\partial y^2} \tag{31}
\]

\[
\frac{\partial^2 \theta_1}{\partial y^2} - \lambda_1^2 \theta_1 = P_r \left( \frac{\partial (\Psi_0, \theta_{0s})}{\partial (y, x)} + \frac{\partial (\Psi_{0s}, \theta_0)}{\partial (y, x)} - 2\Omega E_r \frac{\partial \Psi_0}{\partial y} \frac{\partial \Psi_{0s}}{\partial y} \right) - 2E_r P_r \left( \frac{\partial^2 \Psi_0}{\partial y^2} \frac{\partial^2 \Psi_{1s}}{\partial y^2} + \frac{\partial^2 \Psi_1}{\partial y^2} \frac{\partial^2 \Psi_{0s}}{\partial y^2} \right) \tag{32}
\]

\[
\frac{\partial^2 \omega_{1s}}{\partial y^2} = \frac{\partial (\Psi_{0s}, \omega_{0s})}{\partial (y, x)} - \Omega \frac{\partial^2 \Psi_{0s}}{\partial y^2} \tag{33}
\]

\[
\frac{\partial^2 \theta_{1s}}{\partial y^2} = \frac{\partial (\Psi_{0s}, \theta_{0s})}{\partial (y, x)} - \Omega \left( \frac{\partial \Psi_{0s}}{\partial y} \right)^2 - 2E_r P_r \frac{\partial^2 \Psi_{0s}}{\partial y^2} \frac{\partial^2 \Psi_{1s}}{\partial y^2} \tag{34}
\]

with boundary conditions

\[
\frac{\partial^2 \Psi_1}{\partial y^2} = \frac{\partial^2 \Psi_{1s}}{\partial y^2} = 0, \quad \Psi_0 = \Psi_{1s} = 0, \quad \frac{\partial \theta_1}{\partial y} = \frac{\partial \theta_{1s}}{\partial y} = 0 \quad \text{on} \quad y = 0. \tag{35}
\]

\[
\frac{\partial \Psi_1}{\partial y} = \frac{\partial \Psi_{1s}}{\partial y} = 0, \quad \Psi_1 = \Psi_{1s} = 0, \quad \theta_1 = \theta_{1s} = 0 \quad \text{on} \quad y = S(x). \tag{36}
\]
and so on. Equations (23) to (38) are solved for the stream function $\Psi$, vorticity $\omega$ and temperature distribution $\theta$ and are presented in the appendix. Solutions corresponding to the steady problem and the non-magnetic case are obtained by taking the limits $m \to 0$ and $\Omega \to 0$.

The shear stress at the curved wall $y = b(x)$ is given in [15] by

$$\tau = -\mu \frac{\partial^2 \Psi}{\partial y^2} = \frac{3}{S^2} \frac{\lambda_1^2 \sinh(\lambda_1 S) \text{me}^{it}}{\lambda_1 S \cosh(\lambda_1 S) - \sinh(\lambda_1 S)} + \text{Re} \left( \frac{1}{5} \Omega - \frac{12S}{35S^2} + \omega_1 \text{me}^{it} \right) + O(\text{Re}^2).$$

(40)

From the axial component of the Navier-Stokes equations, we obtain the dimensionless axial pressure gradient as

$$\frac{\partial P}{\partial x} = -\frac{\partial \omega}{\partial y} - \text{Re} \left( \frac{\partial \Psi}{\partial y} \frac{\partial^2 \Psi}{\partial x \partial y} + \frac{\partial^2 \Psi}{\partial \omega} - \Omega \frac{\partial \Psi}{\partial y} \right) - \alpha \frac{\partial^2 \Psi}{\partial \omega \partial y}$$

(41)

On substituting the expression for $\Psi$, this expands to

$$\frac{\partial P}{\partial x} = \frac{3}{S^3} - \left( \frac{\lambda_1^2 \cosh(\lambda_1 S)}{\lambda_1 S \cosh(\lambda_1 S) - \sinh(\lambda_1 S)} \right) \text{me}^{it} + O(\text{Re})$$

(42)

The rate of heat transfer at the wall is given by $Nu = a_0 H/k(T_\infty - T_0)$ where $H = -k \partial T/\partial y$ is the Fourier law of heat conduction. Non-dimensionalising, we obtain $Nu = -\partial \theta/\partial y$. Hence

$$Nu = \frac{3E_c P_r}{S^3} - \frac{6E_c P \lambda_1^2 \text{me}^{it}}{S^3(\lambda_1 S \cosh(\lambda_1 S) - \sinh(\lambda_1 S))^2} \left( \frac{S \lambda_2 \sinh(\lambda_1 S) \sinh(\lambda_2 y)}{(\lambda_1^2 - \lambda_2^2) \cosh(\lambda_2 S)} + \ldots \right)$$

(43)

4. GRAPHICAL RESULTS AND DISCUSSION

The numerical calculations have been performed for the “exponentially” diverging geometry i.e. $S(x) = e^x$, where $x$ is slowly varying. We choose the
pulse $m$ small so that the ensuing flow is a small oscillatory disturbance about the steady flow. We have observed that for every $m$, the Womersley parameter $\alpha$ can be varied only for a range of values, hence we set $\alpha = 0.5$. The effect of varying the Reynolds number $Re$, the Eckert number $Ec$, and the Prandtl number $Pr$ to flow structure is well known in the literature and is not investigated here. Therefore we set $Re = 0.1$, $Ec = 1$ and $Pr = 7.1$. In the ensuing analysis, we assume that $m$ and $\Omega$ can be varied while keeping $\alpha$, $Re$, $Ec$ and $Pr$ fixed. This assumption is valid because of the physics of the problem and the range of values of $m$ and $\Omega$ that are involved.

Fig. 2 shows that the effect of increasing values of $\Omega$ on steady flow is to dampen the velocity profile. This is well known for Hartmann flow. Moreover, for a channel of varying cross section, the dampening is pronounced in the centre of the channel. This creates a stagnation point and consequently fluid is pushed to the walls of the channel thereby increasing the velocity in the boundary layer.

For unsteady flow ($m = 0.01$), equation (45) says that axial velocity is oscillatory about the steady flow axial velocity. Here, $\Omega$ is coupled with the unsteady terms.
in the form \( f(\Omega, \ldots) \cos(t) + g(\Omega, \ldots) \sin(t) \), where functions \( f, g \) are such that \( f > g > 0 \) and both \( f \) and \( g \) are increasing in \( \Omega \). This means that increasing values of \( \Omega \) have the effect of increasing/decreasing axial velocity when axial velocity is increasing/decreasing in time. Figs. 3 and 4 depict this situation for a smaller range of \( \Omega \) values. This suggests that in this model, increasing magnetic force enhances unsteadiness.

The pressure gradient, which is trying to accelerate the fluid is counteracted by the magnetic drag. Hence we observe a pressure gradient drop in the centre of the channel \((y = 0)\) in Fig. 5 for \( t = \pi/4 \). The drop is increasing for increasing values of \( \Omega \). There is no marked qualitative difference for figures of other values of \( t \) in the oscillation. The work being done by the pressure gradient against the magnetic tension is converted to heat. Fig. 6 shows that Ohmic dissipation at the centre of the channel dominates viscous dissipation at the wall for all times \( t \). Increasing values of \( \Omega \) are associated with larger temperature rises. When more and more unsteadiness is introduced into the flow by increasing values of \( m \), the temperature in the center of the channel decreases with increasing \( m \) but still rises.

![Fig. 4. – Axial velocity profile \((m = 0.01, \Omega = 0, 10, 20)\).](image1)

![Fig. 5. – Axial pressure gradient \((m = 0.01, \Omega = 0, 2.5, 5)\).](image2)
with increasing values of $\Omega$. This is depicted in Fig. 7. This suggests that unsteadiness has the effect of cooling the fluid.

Fig. 8 shows that heat transfer takes place upstream from the fluid to the wall, consequently the wall is being heated. The rate of heat transfer is

Fig. 6 – Temperature profile ($m = 0.01, \Omega = 0, 10, 20, R = 0.1, E_c = 1, P_r = 7.1$).

Fig. 7 – Temperature profile ($m = 0.1, \Omega = 0, 10, 20, R = 0.1, E_c = 1, P_r = 7.1$).

Fig. 8 – Rate of heat transfer at the wall ($m = 0.01, \Omega = 0, 10, 20, R = 0.1, E_c = 1, P_r = 7.1$).
decreasing until the adiabatic condition is attained downstream. Increasing values of $\Omega$ and $m$ increase the rate of heat transfer for all times. This means that in the presence of unsteadiness, the magnetic field causes more heat transfer at the wall. This is anticipated from Figs. 6 and 7.

Shear stress at the wall varies with time upstream. Downstream the variations are small as shown in Fig. 9. For steady flow, increasing values of $\Omega$ increase shear stress and shear stress tends to a positive constant that depends on $\Omega$ downstream. For unsteady flow Figs. 10 and 11 show that increasing values of $\Omega$ and $m$ increase shear stress. Prandtl’s separation criterion fails for time dependent flows [10, 15]. However for fixed separation (i.e., separation at a constant location), necessary conditions were proposed in [10]. Among them is the requirement that

$$\int_0^{2\pi} \frac{\partial^2 \Psi}{\partial y^2} (\gamma, 0, t) dt = 0 \quad \text{and} \quad \int_0^{2\pi} \frac{\partial^3 \Psi}{\partial y^3} (\gamma, 0, t) dt < 0$$

with non slip boundary conditions at the walls. From Figs. 10 and 11, we observe that fixed separation is possible only for small values of $\Omega$ and $m$.

Fig. 9. – Wall shear stress $m = 0.1$, $\Omega = 10$.

Fig. 10. – Wall shear stress ($m = 0.01$, $\Omega = 0$, 100, 200).
Fig. 11 – Wall shear stress \( m = 0.1, \; \Omega = 0, 100, 200 \).

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**APPENDIX**

Equations (23) to (38) are solved to obtain the following expressions for \( \Psi \), \( \omega \) and \( \theta \):

\[
\Psi = \frac{3y}{2S} \left( \frac{y^3}{2S^3} + \frac{\lambda_1 y \cosh(\lambda_1 S) - \sinh(\lambda_1 y)}{\lambda_1 S \cosh(\lambda_1 S) - \sinh(\lambda_1 S)} \right) m e^{it} + \\
+ \Re \left( \frac{-3y^7 S_x}{280S^5} + \frac{3y^5 S_x}{40S^5} - \frac{33y^3 S_x}{280S^3} + \frac{3y S_x}{56S} \right) + \Omega S^2 \left[ -\frac{y^5 S_x}{40S^5} + \frac{y^3 S_x}{20S^3} - \frac{y}{40S} \right] + O(Re^2),
\]

(45)

\[
\omega = \frac{3y}{S^3} + \left( \frac{\lambda_1^2 \sinh(\lambda_1 y)}{\lambda_1 S \cosh(\lambda_1 S) - \sinh(\lambda_1 S)} \right) m e^{it} + \Re \left( \frac{S_x}{S^2} \left( \frac{9y^5}{20S^3} - \frac{3y^3}{2S^3} + \frac{99y}{140S} \right) \right) + \Omega \left( \frac{y^3}{2S^3} - \frac{3y}{10S} \right) + O(Re^2),
\]

(46)

\[
\theta = 1 - \frac{3E_e P_e}{4S^6} (y^4 - S^4) + \frac{6E_e P_e \lambda_1^3}{S^3 (\lambda_1 S \cosh(\lambda_1 S) - \sinh(\lambda_1 S))} \cdot \left( \frac{S \sinh(\lambda_1 S) \cosh(\lambda_2 y)}{(\lambda_1^2 - \lambda_2^2) \cosh(\lambda_2 S)} + \frac{2 \cosh(\lambda_1 S) \cosh(\lambda_2 y)}{(\lambda_1^2 - \lambda_2^2)^2 \cosh(\lambda_2 S)} \right) \\
- \left( \frac{y \sinh(\lambda_1 y)}{(\lambda_1^2 - \lambda_2^2)} + \frac{2 \cosh(\lambda_1 y)}{(\lambda_1^2 - \lambda_2^2)^2} \right) m e^{it} + \Re \left( \frac{3P_e^2 S_x y^2}{16S^6} - \right)
\]
\[
\frac{9P_r^2E_cS_x y^8}{224S^10} - \frac{9P_r^2E_c S_x y^2}{8S^5} + \frac{3P_r^2E_c S_x y^6}{40S^9} - \frac{27P_r^2E_c S_x y^8}{560S^10} + \\
+ \frac{3P_r E_c S_x y^6}{1620S^5} - \frac{1011P_r^2E_c S_x y^4}{1120S^2} + \frac{57S_x P_r E_c}{560S^2} + \\
+ \Omega E_c P_r \left( - \frac{7y^6}{10S^6} + \frac{21y^4}{40S^4} - \frac{9y^2}{8S^2} + \frac{31}{40} \right) \theta_1 e^{it} + O(Re^2),
\]

where $S_x = dS/dx$ and $\Psi_1, \omega_1, \theta_1$ are the first order solutions.

REFERENCES