REALISTIC SPIN-ORBIT OPERATORS
FOR THE SUPERASYMMETRIC TWO-CENTER SHELL MODEL

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In order to improve the supersymmetric two-center shell model, shape dependent orbital momentum operators are defined. A realistic dependency of the spin-orbit interaction on the nuclear shape parametrization is obtained and presented in detail. The amelioration produced by this better description is underlined by effecting potential energy surface calculations. The double barrier is obtained along a minimal action integral path in a three-dimensional configuration space. The matrix elements of the spin-orbit interaction are presented in form able to used in numerical codes.

INTRODUCTION

The theoretical study of binary disintegration processes is limited by the difficulties encountered in the calculations of single-particle levels for very deformed one-center potentials. Indeed, on one hand, central potentials are not able to describe in a correct manner the shapes for the passage of one nucleus to two separated nuclei and, on other hand, for very large prolate deformations the sum of single-particle energies obtained from the level scheme reaches an infinite value. These difficulties are overcome by considering that the mean field is generated by nucleons moving in a double center potential. A two-center model allows the description of single-particle energy evolutions from the ground-state up to the formation of two separated fragments [1, 2] of a dinuclear system. Most of the applications of this method are based on a double oscillator potential [3] and more recently, the single-particle motion in fusioning systems and the peripheral collisions were treated employing Wood-Saxon wells [4] with the potential expansion method. A supersymmetric two-center shell model (STCSM) [5], based on the mathematical formalism of Ref. [6] was developed in our institute and was successfully used in investigating cluster emission and alpha-decay [7–10].
The renewal interest for the fission physics determined the use of the STCSM [5] in the analysis of the role played by the single-particle states during the tunneling of the double barrier. This model was subsequently applied to investigate microscopically the role of dissipation [11] on the isotopic mass distribution of fission fragments and the dynamical single particle effects [12, 13] on the neutron-induced cross section. Several interesting results were obtained. It was shown that, apart the effective mass and the deformation energy, the isotopic yields distribution of fission fragments is also managed by the dissipation energy. Also, the rich intermediary resonance structure of the neutron induced fission cross section can be explained by the rearrangement of single-particle orbitals on the way from the ground-state to scission. In fission the necking parameter plays a very important role. The nuclear shapes that characterize the ground-state of the nucleus and the saddle points of the barrier exhibit pronounced necks between the nascent fragments. Investigating fission processes, unfortunately, very large values of the heights of the double barrier were obtained. A cause for this behavior was presumed to be due to the simplified treatment of generalized angular momentum operators. In the following, an improved version of the STCSM is presented that takes into account a realistic dependence of the angular momentum couplings on the shape of the neck. A better description of the double humped barrier is obtained.

MODEL

An axial-symmetric nuclear parameterization is obtained by smoothly joining two intersected spheres of different radii $R_1$ and $R_2$ with a neck surface generated by the rotation of a circle of radius $R_3$ around the symmetry axis, as displayed in Fig. 1. The surface equation is given in cylindrical coordinates:

$$
\rho_c(z) = \begin{cases} 
\sqrt{R_1^2 - (z-z_1)^2}, & z \leq z_{c1}, \\
\rho_3 - s\sqrt{R_3^2 - (z-z_3)^2}, & z_{c1} < z < z_{c2}, \\
\sqrt{R_2^2 - (z-z_2)^2}, & z_{c2} \leq z.
\end{cases}
$$

(1)

Fig. 1 – Nuclear shape parametrization.
The meaning of the geometrical symbols that depends on the shape parameterization can be understood inspecting Fig. 1. This parameterization allows to characterize a single nucleus or two separated nuclei. The macroscopic degrees of liberty of the shape are the elongation characterized by the distance between the centers \( R = z_2 - z_1 \), the necking given by the curvature \( C = s/R^3 \), and the mass-asymmetry denoted \( \eta = R_1/R_2 \).

For the nuclear shape parameterization presented above, the next microscopic potential (in cylindrical coordinates) is appropriate [5]:

\[
V(\rho, z, \varphi) = V_0(\rho, z) + \Delta V_{as}(\rho, z) + \Delta V_{n}(\rho, z) + V_{Ls}(\rho, z, \varphi) + V_{Ls'}(\rho, z, \varphi) - V_c
\]

where the first term \( V_0 \) in the right hand

\[
V_0(\rho, z) = \begin{cases} 
\frac{1}{2} m_0 \alpha_1^2 (z + z_1)^2 + \frac{1}{2} m_0 \alpha_1^2 \rho^2, & z < 0 \\
\frac{1}{2} m_0 \alpha_1^2 (z - z_2)^2 + \frac{1}{2} m_0 \alpha_1^2 \rho^2, & z \geq 0
\end{cases}
\]

represents the two-center semi-symmetric harmonic potential. Such a potential corresponds to a parameterization of the dinuclear system consisting in a sphere of radius \( R_1 \) intersected with a spheroid of semi-axis \( R_1 \) and \( R_2 \) as plotted in Fig. 1. An analytic system of eigenvectors can be obtained for \( V_0 \) by solving the Schrodinger equation:

\[
\left[ -\frac{\hbar^2}{2m_0} + V_0(\rho, z) \right] \Psi(\rho, z, \varphi) = E \Psi(\rho, z, \varphi)
\]

The analytic solution are:

\[
\Psi(\rho, z, \varphi) = Z(z) R(\rho) \Phi(\varphi)
\]

with

\[
\Phi_n(\varphi) = \frac{1}{\sqrt{2\pi}} \exp(im\varphi)
\]

\[
R_{nm}(\rho) = \frac{2n!}{\sqrt{(n+m)!}} \alpha_1 \exp\left(-\frac{\alpha_1^2 \rho^2}{2}\right) (\alpha_1 \rho)^n L_n^m (\alpha_1^2 \rho^2)
\]

\[
Z_n(z) = \begin{cases} 
C_{v1} \exp\left(-\frac{\alpha_2^2 (z + z_1)^2}{2}\right) H_{v1} \left[-\alpha_1 (z + z_1)\right], & z < 0, \\
C_{v2} \exp\left(-\frac{\alpha_2^2 (z - z_2)^2}{2}\right) H_{v2} \left[\alpha_2 (z - z_2)\right], & z \geq 0,
\end{cases}
\]
where \( L_n^m(x) \) is the Laguerre polynomial, \( H_\nu(\zeta) \) is the Hermite function, \( \alpha_i = (m_i\omega_i/\hbar)^{1/2} \) \( (i = 1, 2) \) are length parameters, and \( C_{vi} \) denote the normalization constants. The second term in the right hand of Eq. (2) is a correction that allows the simulation of a shape parameterization given by two intersected spheres of different radii \( R_1 \) and \( R_2 \) as displayed in Fig. 1:

\[
\Delta V_{as}(\rho, z) = V_{as}(\rho, z) - V_0(\rho, z)
\]  

where

\[
V_{as}(\rho, z) = \begin{cases} 
\frac{1}{2} m_0 \omega_1^2 (z + z_1)^2 + \frac{1}{2} m_0 \omega_1^2 \rho^2, & z < z_0' \\
\frac{1}{2} m_0 \omega_2^2 (z - z_2)^2 + \frac{1}{2} m_0 \omega_2^2 \rho^2, & z \geq z_0'
\end{cases}
\]  

is the potential generated by a parameterization given by two spheres of different radii and \( z_0' \) locates the plane on the \( z \)-axis where the two spheres intersect. The third term simulates the existence of a neck region consistent with the Eq. (1)

\[
\Delta V_n(\rho, z) = V_n(\rho, z) - V_{as}(\rho, z)
\]  

where

\[
V_n(\rho, z) = \begin{cases} 
\frac{1}{2} m_0 \omega_1^2 (z + z_1)^2 + \frac{1}{2} m_0 \omega_1^2 \rho^2, & z \leq z_{c1}, \\
\frac{1}{2} m_0 R_1^2 \omega_1^2 + \frac{1}{2} m_0 \left[ \omega_1^2 + (\omega_2^2 - \omega_1^2) \frac{z - z_{c1}}{z_{c2} - z_{c1}} \right], & z_{c1} \leq z \leq z_0', \\
\frac{1}{2} m_0 R_2^2 \omega_2^2 + \frac{1}{2} m_0 \left[ \omega_2^2 + (\omega_2^2 - \omega_1^2) \frac{z_{c2} - z}{z_{c2} - z_{c1}} \right], & z_{c1} \geq z \geq z_0', \\
\frac{1}{2} m_0 \omega_3^2 (z - z_2)^2 + \frac{1}{2} m_0 \omega_3^2 \rho^2, & z \geq z_{c2},
\end{cases}
\]

The contours determined by the potential \( V_n \) are plotted in Fig. 2. In the expression (2), the next two terms are used to add the generalized angular momentum couplings, the spin-orbit interaction

\[
V_{LS}(\rho, z, \varphi) = \begin{cases} 
- \frac{K_1}{\hbar m_0 \omega_{01}}, Ls, & z < z_0', \\
- \frac{K_2}{\hbar m_0 \omega_{02}}, Ls, & z_0' \leq z,
\end{cases}
\]
and the $L^2$ correction. The last term of the right hand is associated to the depths of the two wells.

In a realistic treatment, the spin-orbit interaction operator of a generalized orbital momentum operator:

$$L_S = \frac{1}{2} \left( L^+ s^- + L^- s^+ \right) + L_z s_z$$

depend on a microscopic potential $V_L$ through the following relations:

$$L_z = -i\hbar \frac{\partial V_L}{\partial \rho} \frac{1}{\rho} \frac{\partial}{\partial \phi}$$

$$L^\pm = \mp \hbar \exp(\pm i\phi) \left( \frac{\partial V_L}{\partial \rho} \frac{\partial}{\partial \rho} - \frac{\partial V_L}{\partial z} \frac{\partial}{\partial z} \mp i \frac{\partial V_L}{\partial z} \frac{\partial}{\partial \rho} \right)$$

In the previous version of the STCSM [5], a potential $V_L$ was chosen in order that the angular momentum terms reproduce the energy shifts for two spherical fragments separated at infinity and the those of the initial parent nucleus. The neck dependence of the potential that enters in formulas (13) and (14) is neglected. So, a potential $V_0$ for the angular momentum operator that corresponds only for a two intersected spheres system was used instead $V_L$. This approximation was made in order to avoid cumbersome calculations. Now, we are interested to introduce the neck dependence in the description of the angular momentum operators. For this purpose we define $V_L(\rho, z) = V_0(\rho, z) + \Delta V_n(\rho, z)$.
(ΔV_n being defined by Rel. (9)) as the potential used to construct the generalized angular momentum operator.

After some calculations and rearrangement of terms, the next expressions were obtained for the matrix elements of the \textbf{Ls} interaction components:

\[
\begin{align*}
\langle n' m' s' | \left\{ \frac{-\kappa_i}{\hbar m_0 o_{0i}}, L_z s_z \right\} | n m s \rangle &= -2\hbar m s_z \delta_{m'm} \delta_{s's} \delta_{n'n} \times \\
&\times \left[ k_1 \frac{\alpha_1^2}{\alpha_0} \left( j_{z, v_1, \alpha_z, z_1} + J_1^I (v_1, v_1) \right) + k_2 \frac{\alpha_2^3}{\alpha_0^3} \left( j_{z, v_2, \alpha_z, z_2} - J_2^H (v_2, v_2) \right) \right] + \\
&\times \kappa_2 \frac{\alpha_2^3}{\alpha_0^3} \left( j_{z, v_2, \alpha_z, z_2} - J_2^H (v_2, v_2) \right) \\
\langle n' m' s' | \left\{ \frac{-\kappa_i}{\hbar m_0 o_{0i}}, L^+ s^- \right\} | n m s \rangle &= -\hbar \delta_{m'm+1} \delta_{s' s-1} \times \\
&\times \left[ -k_1 \frac{\alpha_1^2}{\alpha_0} E_1^I - k_1 \frac{\alpha_1^3}{\alpha_0^3} E_1^H + \kappa_2 \frac{\alpha_2^3}{\alpha_0^3} E_2^H \right] \\
&\times \left( \sqrt{n + m + 1} \delta_{n'n} - \sqrt{n \delta_{n'n-1}} \right) - \\
&\times \left[ -k_1 \frac{\alpha_1^3}{\alpha_0^3} E_1^H - \kappa_2 \frac{\alpha_2^3}{\alpha_0^3} E_2^H \right] \times \\
&\times \left( \sqrt{n + m + 1} \delta_{n'n} - \sqrt{n \delta_{n'n-1}} \right) - \\
&\times \left[ -k_1 \frac{\alpha_1^3}{\alpha_0^3} E_2^H - \kappa_2 \frac{\alpha_2^3}{\alpha_0^3} E_3^H \right] \times \\
&\times \left( \sqrt{n + 1} n + m + 1 \delta_{n'n} + (n - 1) \sqrt{n \delta_{n'n-1}} - \sqrt{n(n - 1)(n + m) \delta_{n'n-2}} \right) \\
\end{align*}
\]

(15)

\[
\begin{align*}
\langle n' m' s' | \left\{ \frac{-\kappa_i}{\hbar m_0 o_{0i}}, L^- s^+ \right\} | n m s \rangle &= -\hbar \delta_{m'm-1} \delta_{s's+1} \times \\
&\times \left[ -k_1 \frac{\alpha_1^2}{\alpha_0} E_1^I - k_1 \frac{\alpha_1^3}{\alpha_0^3} E_1^H - \kappa_2 \frac{\alpha_2^3}{\alpha_0^3} E_1^H \right] \\
&\times \left( \sqrt{n + m} \delta_{n'n} - \sqrt{n + 1 \delta_{n'n-1}} \right) - \\
&\times \left[ -k_1 \frac{\alpha_1^3}{\alpha_0^3} E_1^H - \kappa_2 \frac{\alpha_2^3}{\alpha_0^3} E_2^H \right] \times \\
&\times \left( \sqrt{n + m} \delta_{n'n} - \sqrt{n + 1 \delta_{n'n-1}} \right) - \\
&\times \left[ -k_1 \frac{\alpha_1^3}{\alpha_0^3} E_2^H - \kappa_2 \frac{\alpha_2^3}{\alpha_0^3} E_3^H \right] \times \\
&\times \left( \sqrt{n + 1} n + m + 1 \delta_{n'n} + (n - 1) \sqrt{n \delta_{n'n-1}} - \sqrt{n(n - 1)(n + m) \delta_{n'n-2}} \right) \\
\end{align*}
\]

(17)
Here, $s_z$ represents a quantum number for the projection of the nucleon spin. For simplicity, the following abbreviations are used:

\[
E_{I}^{I}(v',v) = C_{v_{I}}C_{v_{I}} \left\{ \frac{1}{2} \hat{J} \left( t_{v_{I}, v_{I}+1, a_{v_{I}}}, -v_{1}, t_{v_{I}, v_{I}-1, a_{v_{I}}} \right) - J_{I}^{I}(v_{I}, v_{I}) - 2v_{1}J_{II}^{I}(v_{I}, v_{1} - 1) - \frac{1}{2} \left[ 1 + F_{p_I}(0) \right] \exp(-\alpha_{2}^{2}z_{1}^{2})H_{v_{I}}(\alpha_{2}z_{1})H_{v_{I}}(-\alpha_{2}z_{1}) \right\}
\]

\[
E_{II}^{II}(v',v) = C_{v_{II}}C_{v_{II}} \left\{ -\frac{1}{2} \left( \hat{J} \left( t_{v_{II}, v_{II}+1, a_{v_{II}}}, -v_{2}, t_{v_{II}, v_{II}-1, a_{v_{II}}} \right) - J_{II}^{II}(v_{II}, v_{II}) \right) + v_{2} \left( \hat{J} \left( t_{v_{II}, v_{II}+1, a_{v_{II}}}, -v_{2}, t_{v_{II}, v_{II}-1, a_{v_{II}}} \right) - J_{II}^{II}(v_{II}, v_{II}) \right) + 2v_{2}J_{II}^{II}(v_{II}, v_{2} - 1) + \frac{1}{2} \left[ 1 + F_{p_{II}}(0) \right] \exp(-\alpha_{2}^{2}z_{2}^{2})H_{v_{II}}(\alpha_{2}z_{2})H_{v_{II}}(-\alpha_{2}z_{2}) - \frac{1}{2} \left[ 1 + F_{p_{II}}(z_{0}) \right] \exp(-\alpha_{2}^{2}z_{2}^{2})H_{v_{II}}(\alpha_{2}z_{2})H_{v_{II}}(\alpha_{2}z_{0} - \alpha_{2}z_{2}) \right\}
\]

\[
E_{III}^{III}(v',v) = C_{v_{III}}C_{v_{III}} \left\{ -\frac{1}{2} \left( \hat{J} \left( t_{v_{III}, v_{III}+1, a_{v_{III}}}, -v_{3}, t_{v_{III}, v_{III}-1, a_{v_{III}}} \right) + v_{3} \left( \hat{J} \left( t_{v_{III}, v_{III}+1, a_{v_{III}}}, -v_{3}, t_{v_{III}, v_{III}-1, a_{v_{III}}} \right) - J_{III}^{III}(v_{III}, v_{III}) \right) - 2v_{3}J_{III}^{III}(v_{III}, v_{3} - 1) + \frac{1}{2} \left[ 1 + F_{p_{III}}(z_{0}) \right] \exp(-\alpha_{3}^{2}z_{3}^{2})H_{v_{III}}(\alpha_{3}z_{3})H_{v_{III}}(-\alpha_{3}z_{3}) \right\}
\]

The symbols with capital letters $J_{i}$ denote integrals along $z$ that are performed numerically using Gauss-Legendre quadratures:
\[
J^I_1(v'_1, v_1) = \int_{z_1}^{0} F_{p_l}(z) [\alpha_1(z + z_1)] \exp\left[-(\alpha_1(z + z_1))^2\right] H_{v'_1}(\alpha_1(z + z_1)) \times \nabla H_{v'_1}(-\alpha_1(z + z_1)) dz \\
J^I_2(v'_1, v_1) = \int_{z_1}^{0} F_{p_l}(z) \exp\left[-(\alpha_1(z + z_1))^2\right] H_{v'_1}(\alpha_1(z + z_1)) \times \nabla H_{v'_1}(-\alpha_1(z + z_1)) dz \\
J^I_3(v'_1, v_1) = \int_{z_1}^{0} F_{p_l}(z) \exp\left[-(\alpha_1(z + z_1))^2\right] H_{v'_1}(\alpha_1(z + z_1)) \times \nabla H_{v'_1}(-\alpha_1(z + z_1)) dz \\
J^I_4(v'_1, v_1) = \int_{z_1}^{0} F_{p_l}(z) \exp\left[-(\alpha_1(z + z_1))^2\right] H_{v'_1}(\alpha_1(z + z_1)) \times \nabla H_{v'_1}(-\alpha_1(z + z_1)) dz \\
J^II_2(v'_2, v_2) = \int_{z_2}^{0} F_{p_l}(z) \exp\left[-(\alpha_2(z + z_2))^2\right] H_{v'_2}(\alpha_2(z + z_2)) \times \nabla H_{v'_2}(-\alpha_2(z + z_2)) dz \\
J^II_3(v'_2, v_2) = \int_{z_2}^{0} F_{p_l}(z) \exp\left[-(\alpha_2(z + z_2))^2\right] H_{v'_2}(\alpha_2(z + z_2)) \times \nabla H_{v'_2}(-\alpha_2(z + z_2)) dz \\
J^II_4(v'_2, v_2) = \int_{z_2}^{0} F_{p_l}(z) \exp\left[-(\alpha_2(z + z_2))^2\right] H_{v'_2}(\alpha_2(z + z_2)) \times \nabla H_{v'_2}(-\alpha_2(z + z_2)) dz \\
J^III_2(v'_2, v_2) = \int_{z_2}^{0} F_{p_l}(z) \exp\left[-(\alpha_2(z + z_2))^2\right] H_{v'_2}(\alpha_2(z + z_2)) \times \nabla H_{v'_2}(-\alpha_2(z + z_2)) dz \\
J^III_3(v'_2, v_2) = \int_{z_2}^{0} F_{p_l}(z) \exp\left[-(\alpha_2(z + z_2))^2\right] H_{v'_2}(\alpha_2(z + z_2)) \times \nabla H_{v'_2}(-\alpha_2(z + z_2)) dz \\
J^III_4(v'_2, v_2) = \int_{z_2}^{0} F_{p_l}(z) \exp\left[-(\alpha_2(z + z_2))^2\right] H_{v'_2}(\alpha_2(z + z_2)) \times \nabla H_{v'_2}(-\alpha_2(z + z_2)) dz \\
J^IV_2(v'_2, v_2) = \int_{z_2}^{0} F_{p_l}(z) \exp\left[-(\alpha_2(z + z_2))^2\right] H_{v'_2}(\alpha_2(z + z_2)) \times \nabla H_{v'_2}(-\alpha_2(z + z_2)) dz \\
J^IV_3(v'_2, v_2) = \int_{z_2}^{0} F_{p_l}(z) \exp\left[-(\alpha_2(z + z_2))^2\right] H_{v'_2}(\alpha_2(z + z_2)) \times \nabla H_{v'_2}(-\alpha_2(z + z_2)) dz \\
J^IV_4(v'_2, v_2) = \int_{z_2}^{0} F_{p_l}(z) \exp\left[-(\alpha_2(z + z_2))^2\right] H_{v'_2}(\alpha_2(z + z_2)) \times \nabla H_{v'_2}(-\alpha_2(z + z_2)) dz \\
\]
The previous expressions determined for angular momentum operator requires the partial derivatives of the correction $\Delta V_n'(p, z)$ with respect the cylindrical coordinates:

$$\frac{\partial \Delta V_n}{\partial\rho} = \begin{cases} m_0\omega^3_0 F_{\rho 1}(z), & z_{c1} < z < z_0', \\ m_0\omega^3_0 F_{\rho 2}(z), & z_0' \leq z < z_{c2}, \end{cases}$$

$$\frac{\partial \Delta V_n}{\partial z} = \begin{cases} m_0\omega^3_0 C_{c1}\rho^2 - m_0\omega^3_0 F_{c1}(z), & z_{c1} < z < z_0', \\ m_0\omega^3_0 C_{c2}\rho^2 - m_0\omega^3_0 F_{c2}(z), & z_0' \leq z < z_{c2}, \end{cases}$$

where

$$F_{\rho 1}(z) = \frac{\omega_2^3 - \omega_1^3}{\omega_1^3} \frac{z - z_{c1}}{z_{c2} - z_{c1}}, \quad F_{\rho 2}(z) = \frac{\omega_1^3}{\omega_2^3} \frac{z_{c2} - z}{z_{c2} - z_{c1}},$$

$$C_{c1} = \frac{\omega_2^3 - \omega_1^3}{2\omega_1^3(z_{c2} - z_{c1})}, \quad C_{c2} = \frac{\omega_1^3}{2\omega_2^3(z_{c2} - z_{c1})},$$

$$F_{c1}(z) = \frac{\omega_2^3 - \omega_1^3}{2\omega_1^3(z_{c2} - z_{c1})} \left[ \rho_3 - s\sqrt{R_3^2 - (z - z_3)^2} \right]^2 +$$

$$\frac{s(z - z_3) \left[ 1 + \frac{\omega_2^3 - \omega_1^3}{\omega_1^3} \frac{z - z_{c1}}{z_{c2} - z_{c1}} \right]}{\sqrt{R_3^2 - (z - z_3)^2}} + (z + z_1)$$

$$F_{c2}(z) = -\frac{\omega_1^3}{2\omega_2^3(z_{c2} - z_{c1})} \left[ \rho_3 - s\sqrt{R_3^2 - (z - z_3)^2} \right]^2 +$$

$$\frac{s(z - z_3) \left[ 1 + \frac{\omega_1^3}{\omega_2^3} \frac{z_{c2} - z}{z_{c2} - z_{c1}} \right]}{\sqrt{R_3^2 - (z - z_3)^2}} + (z - z_2)$$

The relations for the matrix elements of the angular momentum operators were deduced by taking into account the next analytical expressions:

$$\int_0^{\infty} R_{n'm+1}(p) R_{nm}(p) p dp = \frac{1}{\alpha} \sqrt{n + m + 1}\delta_{n'n} - \sqrt{n}\delta_{n'n-1}$$

$$\int_0^{\infty} R_{n'm-1}(p) R_{nm}(p) p dp = \frac{1}{\alpha} \sqrt{n + m}\delta_{n'n} - \sqrt{n + 1}\delta_{n'n+1}$$

$$\int_0^{\infty} R_{n'm+1}(p) \frac{\partial}{\partial p} R_{nm}(p) p dp = \frac{1}{\alpha} \left[ \sqrt{(n + 1)(n + m + 1)(n + m + 2)}\delta_{n'n+1} - \\
- (n + 2)\sqrt{n + m + 1}\delta_{n'n} - (n + m - 1)\sqrt{n}\delta_{n'n-1} + \sqrt{n(n - 1)(n + m)}\delta_{n'n-2} \right]$$
An analytical expression for \( J_{\nu', \nu, \varphi_0} \) can be found in Refs. [5, 6]. To perform the previous expressions, the recurrence relations between the Hermite functions

\[
H_{\nu + 1}(z) = 2zH_{\nu}(z) - 2\nu H_{\nu - 1}(z), \quad \frac{\partial H_{\nu}(z)}{\partial z} = 2\nu H_{\nu - 1}(z),
\]

are also needed. Same results as in Ref. [5] are obtained if all the numerical integrals denoted with capital \( J \) together with the constants \( C_{z_1}, C_{z_2} \) and the functions \( F_{\rho_1}(z) \) and \( F_{\rho_2}(z) \) are set to 0. The previous expressions are obtained in the case \( z_{c_2} > z'_0 \) but can be easily particularized for the case \( 0 < z_{c_1} > z'_0 \) by changing the integral limits. The matrix elements deduced above are used to diagonalize the total Hamiltonian in order to obtain the single-particle levels.

**RESULTS**

An evaluation of the heights of the double humped barrier in fission can be obtained by calculating the deformation energy along the least action trajectory path. The deformation energy is obtained in the frame of the microscopic-macroscopic method. The shell and pairing corrections are added to the liquid drop energy. The macroscopic part is determined within the Yukawa-plus-exponential model extended for binary systems with different charge-densities [14] while the microscopic term is obtained following the Strutinsky prescriptions [15] using the level scheme given by the STCSM. In our previous investigation of the isotopic distribution of fission fragments, the double barriers of three partitions in the mass-asymmetry channel of the parent \( ^{236}\text{U} \) were calculated. The same partitions are analyzed in the actual work using the STCSM improved by taking into consideration the neck dependence of the spin-
orbit interaction. These partitions are: $^{118}$Pd + $^{118}$Pd (a symmetric channel), $^{102}$Zr + $^{134}$Te (an asymmetric channel) and $^{86}$Se + $^{150}$Ce (a very asymmetric channel). An improvement of the results in comparison with those given by the previous version of the STCSM where the neck dependence of angular momentum is completely neglected will be evidenced.

Using the last action principle, it is possible to obtain the paths in the configuration space followed by the fissioning system. The quantum penetrability is

$$P = \exp \left\{ -\frac{2}{\hbar} \int_{R_i}^{R_f} \sqrt{2V(R,C,\eta)B(R,C,\eta,\partial C/\partial R,\partial \eta/\partial R} dR \right\}$$

in the semi-classical Wentzel-Kramers-Brillouin approximation. The two turning points $R_i$ and $R_f$ denote the elongations that characterize the ground-state and the exit point of the barrier, respectively. $V$ is the deformation energy and $B$ is the effective mass along the trajectory. The inertia is computed in the frame of the Werner-Wheeler approximation that means, the flow of the fluid is idealized as non-rotational, non-viscous and hydrodynamic.

It is very difficult to minimize the action integral in a three-dimensional configuration space. Some simplifying assumptions must be introduced. Microscopic approaches to fission [16] claim that the second saddle point is asymmetrical with a value compatible with the observed final mass ratio.

This observation allows us to reduce the number of generalized coordinates in order to make our problem tractable. Therefore, considering that the mass-asymmetry is developed between the first and the second saddle point, the minimization remains to be realized in a two-dimensional configuration space. The minimization is realized as in Ref. [11]. It is assumed that the ratio $R_1/R_2$ varies linearly from unity (first saddle) to the value associated to the final mass partition (second saddle point). To deduce the mass asymmetry in the region of the outer barrier it can be considered that the volume occupied by the light fragment equals the final one. This assumption concerning the variation of the mass asymmetry allows also to simplify the form of the inertia tensor as evidenced in Ref. [11]. The action integral must be minimized now only in a two-dimensional configuration space spanned by $C$ and $R$. The second turning point $R_f$ lies on the equipotential line that characterizes the exit from the outer barrier, that is, $R_f$ is a function of $C$. A simple numerical method can be used to find the paths characterized by different values of $R_f$, associated to some local minima. For that purpose, the function $C = f(R)$ is approximated with a spline function of several variables $C_j (j = 1,n)$ in fixed mesh points $R_j$ located in the interval $[R_i, R_f]$ along the elongation axis. We obtain $C = f(C_j, C_f, R_i, R_f, R_j)$, an interpolation that depends of several parameters $C_j (j = 1,n)$ and the two
turning points \((R_p, C_i)\) and \((R_p, C_f)\) in the configuration space. This interpolation approximates the function \(C = f(R)\), so that, a numerical expression for the functional of the action that depends only on the parameters \(C_j\) is obtained. This expression can be minimized numerically. In our work, the numerical integration is effected within Gauss-Legendre integrals. For different values of \((R_p, C_i)\) and different trial guesses as input data for the minimization, a set of local minima is obtained. The best values for the parameters \((C_i, C_f)\) \((i = 1, n)\) are retained. For the three channels, the paths resemble. In general, the trajectory starts from the ground-state, reaches the region of the second well and change suddenly the slope to penetrate the outer barrier. Between the first and second well, the macroscopic coordinate \(C < 0\), that is, the shapes are swollen in the median region. Penetrating the second barrier, the shapes become necked between the nascent fragments. This behavior is the same as that described in Ref. [11].

The barriers corresponding to the best action trajectories within the previous version of the STCSM are plotted in Fig. 3. The three mentioned partitions are marked with different line types as displayed on the figure.

![Fig. 3 – Theoretical fission barriers of \(^{238}\text{U}\) fission for three mass-asymmetries along the least action trajectory paths. The \(L_s\) coupling is constructed without taking in consideration the neck dependence in the \(V_i\) double center potential.](image)
Phenomenological models [17] that reproduce the fission cross section estimate that the heights of the barrier are about 6 MeV. The theoretical values are unreasonable, being comprised between 12 and 14 MeV. The model succeed however to exhibits some trends that are compatible with experimental findings: the heights of the external barrier are about 1–2 MeV lower for the asymmetric channels than for the symmetric one.

The barriers corresponding to the best action trajectories are plotted in Fig. 4 with the improved version of the STCSM (neck dependence taken into account into the generalized orbital momentum). The same partitions as in Fig. 3 are considered for comparison. It can be observed that the shape of the inner barrier does not change with respect the asymmetry. The outer barrier for the symmetric split is larger while that of the very asymmetric channel is very thin. Including the neck correction in the generalized angular momentum, the heights becomes more reasonable (about 8 MeV) and much closer on experimental findings given in Ref. [17]. The inner barrier is 6 MeV in Ref. [11] while in the actual calculations decreases to about 5.8 MeV. The values for the barrier heights extracted from experimental observations using appropriate statistical models are for the same $^{236}$U parent [11] 5.6 and 5.5 MeV for the inner and outer barriers, respectively.

![Fig. 4. – Same as Fig. 3 when the STCSM takes into consideration the neck dependence in the angular momentum operator.](image)
It can be concluded, that introducing the neck correction in the generalized angular momentum for the $Ls$-interaction, the model becomes more realistic.

The model is subject to some intrinsic limitations that can not be overcome. The following observations must be kept in mind when the formalism is applied. The STCSM can only describe the trends of the deformation energy. Values of the heights of the barrier that match precisely the experimental findings can not be obtained due to the following reasons: \textit{(i)} The Nilsson coefficients values $\kappa$ and $\mu$ are obtained by fitting the behavior of some orbitals lying around the Fermi energy. Different values for these coefficients must be used to reproduce the behavior of levels far from the Fermi energy; \textit{(ii)} The potential used to define $V_L$, entering in the generalized angular momentum, behaves as an oscillator on the nuclear surface. A real potential is more abrupt on the surface as shown by the more realistic Wood-Saxon potential. This a model deficiency. This deficiency is removed by introducing a $L_2^2$ term as a correction to the harmonic potential. For consistency, this term must be also present in the definition of $V_L$. Unfortunately, the present formalism does not allow the presence of the $L_2^2$-correction in $V_L$. Due to these limitations, very accurate values for the heights of the double barrier are not expected.

In Refs. [18, 19] a huge number of configurations in a multidimensional configurations space were analyzed and all possible local minima were scanned. It was show that the lowering of barrier heights requires that microscopic-model parameters be redefined. The same microscopic parameters are employed in this work, so an improvement of results given by the STCSM is also expected with a better set of parameters. It was also claimed that results from break-up of the nucleus have already established their identity at the saddle point, assertion in accordance with the approximation used in the present work to obtain the best action trajectories.

The STCSM is ameliorated having as purpose to investigate the main trends of the fission process. However, in most treatments of disintegration processes, the mean behavior of the phenomenon are determined by the rearrangements of orbitals located in the Fermi energy region. For example, the resonances in the intermediate structure of the fission cross-section are determined mainly by low energy excitations that allow high macroscopic kinetic energies and therefore high yield in the cross-section. The parameters $\kappa$ and $\mu$ are appropriate to this region, so that, the energy shifts exhibited by these levels as function of the generalized coordinates are of enough confidence. In this context, this model can be used to investigate single-particle effects during the nuclear motion from ground-state up to separation into two fragments.

REFERENCES

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