The forced flow of an electrically conducting viscous incompressible fluid, due to an infinite impervious rotating disk bounded by porous medium has been investigated. A uniform magnetic field is applied in the direction normal to the flow. It is assumed that the flow between the disk and the porous medium is governed by Navier-Stokes equations and that in the porous medium by Brinkman equations. Flows in the two regions are matched at the interface by assuming that the velocity and stress components are continuous at it. At the interface (porous medium-clear fluid boundary), a modified set of boundary conditions suggested by Ochoa-Tapia and Whittaker is applied. Analytical expressions for the velocity and shearing stress are calculated and effects of various parameters upon them are examined.

Key words: Porous medium, MHD, rotating disk.

INTRODUCTION

The requirements of modern technology have stimulated the interest in fluid flow studies, which involve the interaction of several phenomena. One such study is presented, when a viscous fluid flows over a porous surface, because of its importance in many engineering problems such as flow of liquid in a porous bearing (Joseph and Tao [1]), in the field of water in river beds, in petroleum technology to study the movement of natural gas, oil and water through the oil reservoirs, in chemical engineering for filtration and purifications process. Cunningham and Williams [2] also reported several geophysical applications of flow in porous medium, viz. porous rollers and its natural occurrence in the flow of rivers through porous banks and beds and the flow of oil through underground porous rocks.

The mathematical theory of the flow of fluid through a porous medium was initiated by Darcy [3]. For the steady flow, he assumed that viscous forces were in equilibrium with external forces due to pressure difference and body forces. Later on Brinkman [4] proposed modification of the Darcy’s law for porous medium. In most of the examples, the fluid flows through porous medium, have
two regions. In region I, the fluid is free to flow and in region II, the fluid flows through the porous medium. To link flows in two regions certain, matching conditions are required at the interface of two regions. This type of couple flow, with different geometry and with several kinds of matching conditions, have been examined by several authors, viz. William [5] and Ochoa-Tapia et al. [6–7]. Srivastava et al. [8] discussed the flow and heat transfer of a viscous fluid confined between a rotating plate and a porous medium, by assuming that the flow in the porous medium was governed by Brinkman equation [4] and that in the free flow region by the Navier-Stokes equations. The problem (in which the liquid occupies the semi-infinite region on one side of the disk and the motion is axially symmetric) of steady forced flow of an incompressible viscous fluid against a rotating disk was studied by Schlichting et al. [9]. A complete review of this paper and also same related work has been given by Moore [10]. Recently, Chaudhary et al. [11] discussed the flow of viscous incompressible fluid confined between a rotating disk and a porous medium.

The subject of hydromagnetics has attracted the attention of many authors, due not only to its own interest, but also to many applications to the problems of geophysical and astrophysicals significance. It is desirable to extend many of the available viscous hydrodynamic solutions to include the effects of magnetic field, for those cases when the viscous fluid is electrically conducting. In view of its wide applications in industrial and other technological fields, the problem of flow near a rotating disk has been extended to hydrodynamics initially by Sparrow et al. [12] and Katukani [13]. Kumar et al. [14] and Watanabe et al. [15] studied MHD flow near a rotating disk. The computational analysis of MHD flow near a rotating disk studied by Ariel [16].

The present investigation is concerned the forced flow generated within a porous medium by a rotating disk near it. The gap between them is filled with an incompressible electrically conducting fluid. The formulation is developed and applied to obtain the variations in the velocity and shearing stress as these evolve in general flow, induced by a rotating disk in the presence of magnetic field.

**FORMULATION OF THE PROBLEM**

We consider the steady flow of an incompressible viscous fluid confined between a rotating disk and a porous medium fully saturated with the fluid. Let \((r^*, \theta^*, z^*)\) be the set of cylindrical polar coordinates and let the disk rotates with angular velocity \(\Omega\) about an axis \(r^* = 0\) and be represented by \(z^* = d\). A uniform magnetic field \(\mathbf{B}\) is applied in the direction normal to flow. The problem we consider here may be represented geometrically by Fig. 1. We assume that the magnetic Reynolds’s number is small so that the induced magnetic field can be neglected. Conductivity of the fluid is not very large. Since, no external electric
field is applied and the effect of polarization of ionized fluid is negligible, it can be assumed that the electric field is zero. The region \( z^* \leq 0 \), which is filled with the porous material fully saturated with the liquid. The region \( 0 \leq z^* \leq d \) is called region I, the region \( z^* \leq 0 \) is called region II and \( z^* = 0 \) is the interface between the two regions.

The Navier-Stokes equations and continuity equation in region I, are given by

\[
\begin{align*}
\rho u^* u^* + w^* u^*_z - v^* v^*_r/ r^* &= - p^*_r + \mu (\nabla^2 u^* - u^*/ r^2) - \sigma B^2 u^*, \\
\rho v^* v^* + w^* v^*_z - u^* v^*_r/ r^* &= - p^*_\theta/ r^* + \mu (\nabla^2 v^* - v^*/ r^2) - \sigma B^2 v^*, \\
(r^* u^*)_r/ r^* + w^*_z &= 0,
\end{align*}
\]

where \( \rho \) is the density, \( p \) is the pressure, \( \sigma \) is the electrical conductivity and \( B \) is the intensity of magnetic field. The velocity components are \( u^*, v^*, w^* \) in the \( r^*, \theta^*, z^* \) direction, respectively. The porous region \( z^* < 0 \) is called region II and in this region the flow is governed by Brinkman [4] equations. These with the continuity equation are given by

\[
\begin{align*}
-P^*_r + \mu_e (\nabla^2 \U^* - \U^*/ r^2) - \mu \U^*/ k - \sigma B^2 \U^* &= 0, \\
-P^*_\theta/ r^* + \mu_e (\nabla^2 \V^* - \V^*/ r^2) - \mu \V^*/ k - \sigma B^2 \V^* &= 0,
\end{align*}
\]
\[ \frac{(r^* U^*)}{r^*} + W_z^* = 0, \]  

(6)

where \( k \) is the permeability of the porous medium, \( \mu_e \) is the effective viscosity for Brinkman flow model, which is different from \( \mu \), the viscosity of the fluid, \( P^* \) is the pressure in porous medium region and \( U^*, V^*, W^* \) be the velocity components in porous medium at \( r^*, \theta^*, z^* \) directions, respectively. Givler and Altobelli [17] have determined experimentally \( \mu_e \) for steady flow through a wall-bounded porous medium and their result shows that, \( \mu_e = (7.5^{1.34}) \mu \). The boundary conditions of the problem are

\[ u^* = ar^*, \quad v^* = r^* \Omega, \quad w^* = 0, \quad \text{at} \quad z^* = d, \]  

(7)

\[ U^* \to 0, \quad V^* \to 0, \quad \text{as} \quad z^* \to -\infty. \]  

(8)

We use the matching condition at the interface as suggested by Ochao-Tapia and Whittaker [6–7]. These conditions which are investigated theoretically and experimentally which state that equation requires a discontinuity in the shear stresses, while, retaining the continuity of the velocity. The steady fully developed laminar flow in the parallel plate and cylindrical channel partially filled with a porous medium and partially with clear fluid, using the matching conditions of [6–7] investigated by Kuznetsov [18]. Using these conditions Srivastava [19] has also discussed the flow of a viscous fluid confined between a torsionally disk and a porous medium fully saturated with the liquid. At the interface of the porous medium and clear fluid \( z^* = 0 \), we assume the velocity components and pressure are continuous and the jumps in shearing stresses \( \tau_{z\theta} \) and \( \tau_{zr} \) as given by Ochao-Tapia and Whittaker [6–7].

These assumptions in our notation can be written as:

\[
\begin{align*}
\mu_e U_z^* - \mu u_z^* &= \beta \mu U^*/\sqrt{k}, \\
\mu_e V_z^* - \mu v_z^* &= \beta \mu V^*/\sqrt{k}, \quad \text{at} \quad z^* = 0
\end{align*}
\]  

(9)

**EQUATION OF MOTION**

We assume the following form of velocity components for region I

\[
\begin{align*}
&u^* = r^* f'(y), \quad v^* = r^* \Omega g(y), \quad w^* = -2d \Omega f(y), \\
&\rho^* = -\mu \Omega p_1(y), \quad y = z^*/d,
\end{align*}
\]  

(10)

where prime denotes differentiation with respect to \( y \). This form of the velocity components satisfied the equation of continuity (3). Substituting equation (10) in
equations (1) and (2), we get the following equations of motion in the direction of \( r \) and \( \theta \), respectively:

\[
R \left[ (f'^2 - 2f'f'' - g') \right] = f'''' - M^2 f',
\]

(11)

\[
2R \left[ f'g' - fg'' \right] = g'' - M^2 g,
\]

(12)

where \( R \) (Reynolds number) = \( \rho \Omega d^2/\mu \), \( M \) (Hartmann number) = \( \sqrt{\frac{\sigma B^2 d^2}{\mu}} \).

The equation in the direction of \( z \) serves merely to determine the axial pressure gradient and hence is not given. Assume the following form of velocity components for region II

\[
U^* = r^* \Omega^* F(y), \quad V^* = r^* \Omega^* G(y), \quad W^* = -2d^* \Omega^* F(y), \quad P^* = -\mu \Omega^* P(y)
\]

(13)

The forms of the velocities in equation (13) are so chosen that the equation of continuity (6) is satisfied. Substituting equation (13) in equations (4) and (5), we get the following equations in directions of \( r \) and \( \theta \), respectively:

\[
\gamma^2 F'''' - (\sigma^2 + M^2) F' = 0,
\]

(14)

\[
\gamma^2 G'''' - (\sigma^2 + M^2) G = 0,
\]

(15)

where \( \sigma \) (Darcy number) = \( d/\sqrt{k} \), \( \gamma^2 = \mu_s/\mu \).

Boundary conditions (7), (8) and (9) at interface can be written as:

\[
f = 0, \quad f' = S, \quad g = 1, \quad \text{at} \quad y = 1,
\]

(16)

\[
F' \to 0, \quad G \to 0, \quad \text{as} \quad y \to -\infty,
\]

(17)

\[
\begin{align*}
f &= F, \quad f' = F' \cdot \gamma^2 F'' - f'' = \beta \sigma F', \quad \text{at} \quad y = 0 \\
g &= G, \quad \gamma^2 G' - g' = \beta \sigma G, \quad \text{at} \quad y = 0
\end{align*}
\]

(18)

where \( S = a/\Omega \) is dimensionless forced parameter assumed to be small (\( S \leq 1 \)).

**SOLUTION OF THE PROBLEM**

The solution of equation (14) and (15) satisfying the boundary conditions (17) are given by:

\[
F'(y) = A e^{\alpha y}, \quad F(y) = (A/\alpha) e^{\alpha y} + C, \quad G(y) = B e^{\alpha y},
\]

(19)

where \( \alpha = \sqrt{\frac{\sigma^2 + M^2}{\gamma^2}} \).
The constants A, B and C of integration can be determined from matching conditions (18). In our present effort, we make the small Reynold’s number approximations to the viscous equations. We consider the distance d between the rotating disc and porous interface as small, hence Reynold’s number may be also taken small. For small values of R, a regular perturbation scheme can be developed for equations (11) and (12) by expanding f and g in powers of R as:

\[ f = \sum_{n=0}^{\infty} R^n f_n, \quad g = \sum_{n=0}^{\infty} R^n g_n. \]  

(20)

As f and g have to be matched with equation (19) at the interface, the constants A, B and C must also be expanded in power of R as:

\[ A = \sum_{n=0}^{\infty} R^n A_n, \quad B = \sum_{n=0}^{\infty} R^n B_n, \quad C = \sum_{n=0}^{\infty} R^n C_n. \]  

(21)

Using this perturbation scheme, the solutions of equations (11) and (12) for Region I are given by:

\[ f'(y) = a_3 e^{My} + a_4 e^{-My} + R \left[ (k_1 - d_{15}y) e^{My} + (k_2 + d_{16}y) e^{-My} - d_{13} e^{2My} + d_{14} e^{-2My} - d_{17} \right], \]  

(22)

\[ f(y) = \frac{a_3}{M} e^{My} - \frac{a_4}{M} e^{-My} + a_5 + R \left[ \frac{k_1}{M} + \frac{d_{15}}{M^2} - \frac{d_{15}}{M} y \right] e^{My} - \left( \frac{k_2}{M} + \frac{d_{16}}{M^2} + \frac{d_{16}}{M} y \right) e^{-My} - \frac{d_{13}}{2M} e^{2My} - \frac{d_{14}}{2M} e^{-2My} - d_{17}y + k_3, \]  

(23)

\[ g(y) = a_1 e^{My} + a_2 e^{-My} + R \left[ (h_1 + d_{6}y) e^{My} + (h_2 - d_{5}y) e^{-My} - d_{4} \right], \]  

(24)

and solutions of equation (19) in porous medium are given by:

\[ F'(y) = e^{\alpha y} \left[ (a_3 - a_4) \frac{M}{\alpha} \right] + R e^{\alpha y} \left[ \frac{1}{\alpha} (k_1 M - k_2 M - 2M d_{13} - 2M d_{14} - d_{15} + d_{16}) \right], \]  

(25)

\[ F(y) = e^{\alpha y} \left[ (a_3 - a_4) \frac{M}{\alpha \alpha} \right] - a_3 \left( \frac{M}{\alpha \alpha} - \frac{1}{M} \right) + a_4 \left( \frac{M}{\alpha \alpha} + \frac{1}{M} \right) + a_5 + R \left[ \frac{e^{\alpha y}}{\alpha \alpha} d_{20} + \frac{1}{2M^2} (2k_1 M - 2k_2 M - d_{13} M - d_{14} M + 2d_{15} - 2d_{16} + 2M^2 k_3) - \frac{1}{\alpha \alpha} d_{20} \right], \]  

(26)

\[ G(y) = 2M e^{\alpha y} \left[ (a + M) e^{My} + (M - a) e^{-My} \right] + Re^{\alpha y} \left[ \frac{1}{\alpha} (h_1 M - h_2 M - d_{5} + d_{6}) \right], \]  

(27)
where

\[ \alpha = \frac{\sqrt{\sigma^2 + M^2}}{\gamma^2}, \quad a = \gamma^2 \alpha - \beta \sigma, \quad a_1 = (a + M)/[e^M(a + M) + e^{-M}(m - a)], \]

\[ a_2 = (M - a)/[e^M(a + M) + e^{-M}(m - a)], \]

\[ a_3 = S(M + a)/[e^M(M + a) + e^{-M}(M - a)], \]

\[ a_4 = S(M - a)/e^M(M + a) + e^{-M}(M - a), \quad a_5 = (a_4 e^{-M} - 3a_3 e^M)/M, \]

\[ d_1 = 4a_2 a_3 + 4a_3 a_4, \quad d_2 = 2a_2 a_5 M, \quad d_3 = -2a_1 a_3 M, \quad d_4 = d_1 / M^2, \]

\[ d_5 = d_2 / 2M, \quad d_6 = d_3 / 2M, \quad d_7 = d_5 - d_6 - ad_4, \]

\[ h_1 = [d_4 (M + a) + d_5 (M + a) e^{-M} - d_6 (M + a) e^M - h_3 (M + a) e^M] / \]

\[ [e^M(M + a) + e^{-M}(M - a)] + h_3, \]

\[ h_2 = [d_4 (M - a) + d_5 (M - a) e^{-M} - d_6 (M - a) e^M - d_7 e^M] / \]

\[ [e^M(M + a) + e^{-M}(M - a)], \]

\[ h_3 = d_7 / (M - a), \quad d_8 = a_4^2 + 3a_3^2, \quad d_9 = a_5^2 + 3a_4^2, \quad d_{10} = 2a_3 a_3 M, \quad d_{11} = 2a_4 a_3 M, \]

\[ d_{12} = 2a_1 a_2 + 2a_3 a_4, \quad d_{13} = d_8 / 3M^2, \quad d_{14} = d_9 / 3M^2, \quad d_{15} = d_{10} / 2M, \]

\[ d_{16} = d_{11} / 2M, \quad d_{17} = d_{12} / M^2, \]

\[ d_{18} = d_{13}(2M - a) + d_{14}(2M + a) + d_{15} - d_{16} - ad_{17}, \]

\[ d_{19} = d_{13} e^{2M} - d_{14} e^{-2M} + d_{15} e^M - d_{16} e^{-M} + d_{17}, \]

\[ k_1 = (M^2 - a^2) d_{19} / [e^M(M + a) + e^{-M}(M - a) + d_{18} e^M] + d_{18} / (M - a), \]

\[ k_2 = (M - a) d_{19} / [e^M(M + a) + e^{-M}(M - a) + d_{18} e^M], \]

\[ k_3 = [e^M(2d_{15} M - 2d_{15} - 2k_1 M) + e^{-M}(2k_2 M + 2d_{16} M + 2d_{16}) + d_{13} M e^{2M} + \]

\[ + d_{14} M e^{-2M} + 2M^2 d_{17}]/2M^2, \]

\[ d_{20} = k_1 M - k_2 M - 2M d_{15} - 2M d_{14} - d_{15} + d_{16}. \]

Now upon calculating the velocity fields, we can calculate the shearing stresses at the rotating disk and given as:

\[ [\tau_{rz}]_{z=1} = \frac{\mu \Omega r}{M} f''(y) = \frac{\mu \Omega r}{M} f''(1), \quad (28) \]

where
\[ f''(l) = a_3 M e^M - a_4 M e^{-M} + R [M(k_1 - d_{15}) e^M - d_{15} e^M - M (k_2 + d_{16}) e^{-M} + d_{16} e^{-M} - 2M d_{13} e^{2M} - 2M d_{14} e^{-2M}], \]

and

\[ [\tau_{x\theta}]_{l=1} = \frac{\mu \Omega r}{d} g'(l), \quad (29) \]

where

\[ g'(l) = a_1 M e^M - a_2 M e^{-M} + R [M(h_1 + d_6) e^M + d_6 e^M - M (h_2 - d_5) e^{-M} - d_5 e^{-M}]. \]

**DISCUSSION**

The study of the velocities and shearing stress of the MHD flow of a viscous, incompressible, electrically conducting fluid through a porous medium induced by an impervious rotating disk has been analysed in the preceding sections. This enables us to carry out the numerical computations for the velocities and shearing stress at the rotating disk for various values of the Hartmann number (M), ratio of viscosity (\( \gamma^2 \)), forced parameter (S) and Darcy number (\( \sigma \)). The velocity components in a porous medium against the distance from the interface (–y) have been plotted in Figs. 2, 3 and 4 for various values of parameters which are consistent with Givler and Altobelli [17] and Srivastava [19].

![Graph](image-url)

**Fig. 2 – The radial (F') velocity components in porous medium for \( \beta = 0.5 \) and \( R = 0.2 \).**
Fig. 3 – The axial (–F) velocity components in porous medium for $\beta = 0.5$ and $R = 0.2$.

Fig. 4 – The transverse (–G) velocity components in porous medium for $\beta = 0.5$ and $R = 0.2$. 
Fig. 2 reveals that the radial velocity components \( (F') \) are maximum at the interface and decay exponentially as we enter the porous medium, vanishing at a large distance from the interface. It is observed that the radial velocity component increases with increasing magnetic parameter \( (M) \) and forced parameter \( (S) \) but decreases with increases of \( \sigma \). The magnitude of radial velocity falls sharply with increase in Darcy number. We conclude that when the magnetic field is strong, the radial velocity in the porous medium increases. The depth of penetration is greater for \( \gamma^2 = 1 \) as compared to that for \( \gamma^2 = 5 \). The axial velocity component in porous medium has been shown in Fig. 3. We have drawn \( -(\mathbf{F}) \) against the distance from the interface for \( R = 0.2 \) and \( \beta = 0.5 \), taking the different values of \( \gamma^2, M, S \) and \( \sigma \). It is observed that the axial velocity component at a large distance from the interface does not vanish. A boundary layer is formed at the interface whose thickness is reduced with an increase of \( \sigma \) and attains a constant value. It decreases with an increase of \( M \) and \( \sigma \) but increases with an increase of \( \gamma^2 \) and \( S \). By increasing the large parameter, it’s magnitude increases sharply. It is concluded that the rotation of a disk near a porous medium fully saturated with the fluid, extracts the fluid from the porous medium. This fact may be used by geologists to extract fluids from the porous ground or rocks. We have plotted the graph of transverse velocity \(-\mathbf{G})\) in porous medium (Fig. 4) against the distance from the interface for \( R = 0.2 \) and \( \beta = 0.5 \), taking the different values of \( \gamma^2, M, S \) and \( \sigma \). It is observed that the transverse velocity decreases exponentially as we enter the porous medium. It decreases with increase in both \( M \) and \( \sigma \), while, reverse effect is observed for \( \gamma^2 \) and \( S \). Further, we observed that when the magnetic field is strong, the transverse velocity in the porous medium decreases. It is concluded that the flow in the porous medium in the transverse direction reaches maximum value at the interface and decays exponentially as we enter inside the porous medium, vanishing as \( y \to \infty \).

In Table 1, we have presented the values of shearing stress components at the rotating disk. The table shows that they increase with an increase in \( \gamma^2, S \) and \( M \) but decrease with increasing \( \sigma \). Further, it is observed that if we take \( \gamma^2 = 1, \beta = 0, \)

<table>
<thead>
<tr>
<th>( \gamma^2 )</th>
<th>( S )</th>
<th>( M\sigma )</th>
<th>( f^*_1(1) )</th>
<th>( g^*(1) )</th>
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</thead>
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<tr>
<td>2</td>
<td>0.5</td>
<td>0.5</td>
<td>0.2101</td>
<td>0.2029</td>
</tr>
<tr>
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<td>0.7</td>
<td>0.5</td>
<td>0.3852</td>
<td>0.3776</td>
</tr>
<tr>
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<td>0.5</td>
<td>0.5</td>
<td>1.4160</td>
<td>1.4048</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.8</td>
<td>0.3291</td>
<td>0.3223</td>
</tr>
</tbody>
</table>
MHD forced flow of a conducting viscous fluid

M = 0 and S = 0 in our analysis and $\phi = \lambda = 1$, $\alpha = 0$ in the work of Srivastava and Barman [20], the results of both the studies are comparable. Further, it is noted that if we take M = 0 and S = 0 in our analysis, the results reduced to that of Chaudhary et al. [11].

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