FILTERING SHOCKS IN DISCRETE APPROXIMATIONS TO HYPERBOLIC CONSERVATION LAWS*

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Following [1] and [6] in this paper we design proper filters to recover the exact solution to problems involving weak discrete stationary shocks for system of conservation laws.

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1. INTRODUCTION

Let us consider a system of conservation laws

\[ u_t + f(u)_x = 0, \quad (1) \]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a smooth vector function and the unknown function \( u(x, t) \in \mathbb{R}^n \) and \(-\infty < x < +\infty, \quad t > 0\).

These equations are of great practical importance since they govern a variety of physical phenomena that appear in fluid mechanics, astrophysics, groundwater flow, meteorology semiconductors, and reactive flows. We are interested to study the solution of (1) in the case of stationary shocks. By stationary shock we understand the solution given by \( u = u_L, \quad x < x_0 \) and \( u = u_R, \quad x > x_0 \), where \( u_L \) and \( u_R \) satisfy the Rankine-Hugoniot condition \( f(u_L) = f(u_R) \).

Equation (1) is approximated using a dissipative system

\[ u_t + f(u)_x = -(i\partial_x)^k \left[ A(u)(i\partial_x)^\frac{k+1}{2} \right] u, \quad (2) \]

where \( k \) must be odd. It is known that stationary solutions \( u_{st} \) of (2) which attain the above limits \( u_L \) and \( u_R \) as \( x \rightarrow -\infty \) and \( x \rightarrow +\infty \), correspondingly are called


stationary viscous shocks. Moreover, if the difference $|u_R - u_L|$ is small, then the shock is called weak. Using [3] and some other assumptions it is known that we have existence of the viscous shocks for $k = 1$ and also for $k = 3$. Under some assumptions of distinct eigenvalues for the differential of the function $u$ and entropy conditions we have the existence of viscous shocks for $k = 1$ and $k = 3$ follows from [3]. The most accepted numerical schemes for shock wave computations usually produces monotone shock profiles. In this paper we consider high-order schemes which are known to converge exponentially fast to the stationary solution. Numerical solutions obtained using high-order schemes are post processed using filters. In the next section we describe briefly a procedure of post-processing of the numerical solution using filters based on the gradient of the numerical approximation.

2. FILTERS

In order to construct filters to post process the solution the main ingredient we use is the gradient of the numerical approximation to the scalar conservation law (1). This gradient is used to detect the region of jump discontinuity. If a mesh point is detected to have a large gradient then we reached the region of the jump discontinuity. This procedure of post-processing take in account the conservation. Next, we describe briefly the post-processing of a simple shock.

2.1. SIMPLE SHOCK

Let us consider for illustration purposes the Burgers equation

$$u_t + \left( \frac{u^2}{2} \right)_x = 0, \quad (3)$$

where $-1 < x < 1$ and $t > 0$, with initial conditions given as

$$u(0, t) = u_0(x) = \begin{cases} +1, & x < 0 \\ -1, & x > 0 \end{cases},$$

which represents a simple strong shock. Numerical approximation obtained using high order Runge-Kutta as in [1] is shown in Fig. 1. It is clear that the profile of the numerical solution is smeared, but we can spot a left state $u_L$, which can be identified for values $x < 0$, and, respectively a right state $u_R$ which can be identified for values $x > 0$. To filter out the shock we consider the integral of the numerical solution as the sum of all values $u_t$ over the whole domain.
Fig. 1 – Numerical solution of Burgers equation for a simple shock.

The filtered solution is a step function which is discontinuous at the origin from value +1 to value −1. Firstly we determine the region where the gradient is large, to find exactly the points $x_L$ and $x_R$ which represents the most left point from where the numerical solution start to present a large gradient, $x_R$ and is the most right point with the same property. To calculate the gradient of the numerical solution we use

$$\frac{u(x_{i+1}) - u(x_i)}{h},$$

where $i = 1, \ldots, n$ represents the computational mesh. Due to the particular form of the shock, the next step in the construction of the filter is to find the middle point of the discontinuity region. In a straightforward manner we approximate the integral of the numerical solution using a simple sum of the values $u_i$ over the whole domain. There exist one requirement of conservation law which must be fulfilled. We choose to preserve the density equations. Furthermore, following again the numerical approximation of the exact solution, on the left hand side of the point $x_L$ and in the right hand side of the point $x_R$ we determine first a left state $u_L$ and, respectively a right state $u_R$. These values will replace the numerical solution obtained using a high-order method in points where we have detected the large gradient. The next step is to replace the points where the numerical solution have a big gradient in the following way: from the left to the middle point with the left state determined above $u_L$ and from the middle point to the $x_R$ most right point from where the solution start to show evidence of a large gradient, and then we replace the points of the profile by $u_R$, the right state. In order to satisfy the conservation law we put together the difference between the sum of values of the initial profile obtained by the numerical multi-step method
like Adams-Bashforth or Runge-Kutta and the sum of the values of the filtered profile. This difference is added to the filtered profile to preserve the integral from the conservation law. The procedure to filter the contact discontinuity is similar to the procedure to filter a simple shock. In the next section we describe briefly the procedure to filter the rarefaction wave.

2.2. RAREFACTION WAVE

First let us suppose that \( u_L \) and \( u_R \) are two states that violate the entropy condition \( \lambda(u_L) > \lambda(u_R) \) and consequently we have a weak solution that violates the entropy condition. In this case the solution is an expansion or rarefaction wave. The solution is self similar and depends on \( \frac{x}{t} \) and has the general form

\[
    u(x, t) = w\left(\frac{x}{t}\right) = \begin{cases} 
    u_L, & \frac{x}{t} \leq \lambda(u_L) \\
    u\left(\frac{x}{t}\right), & \lambda(u_L) \leq \frac{x}{t} \leq \lambda(u_R), \\
    u_R, & \frac{x}{t} \geq \lambda(u_L) 
\end{cases}
\]

where \( w \) is not necessary a linear function. Without losing generality for the numerical example considered in this paper we approximate the function \( w \) using a linear function. For this we need to know what mean a left state, a right state

![Post processed rarefaction wave for Burgers equation.](image)
and the slope between them. A first example to filter the rarefaction wave is given using the numerical solution obtained for Burgers equation (3). The solution is known to be an approximate linear function between points of discontinuity. In Fig. 2 we present the solution filtered for a rarefaction wave formed from values between 1 and −1.

### 3. NUMERICAL EXAMPLES

This example comes from a classical problem in gas dynamics. The shock-tube problem is a very interesting test case because the exact time-dependent solution is known and can be compared with the solution computed applying numerical discretisations. In this case we apply the Runge-Kutta scheme to the one dimensional system,

\[ u_t + f(u)_x = 0, \]

where

\[ u = (\rho, \rho u, E)^T, \]

and

\[ f(u) = (\rho u, P + \rho u^2, u(E + P)), \]

augmented with \( P = (\gamma - 1) \left( E - \frac{1}{2} \rho u^2 \right) \). Here \( \rho \) represents the density, \( u \) represents the velocity and \( E \) the total specific energy. The initial data is in the Riemann form

\[ u(x, 0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0. \end{cases} \]

The first Riemann problem was proposed by Sod [7]. We denote \( m = \rho u \) we have respectively function the vector \( u \) in the form

\[ u = (\rho, m, E)^T, \]

and

\[ f(u) = \left( m, P + \frac{m^2}{\rho}, \frac{m}{\rho}(E + P) \right). \]

Following [8] we say that the initial solution of the shock-tube problem is composed by two uniform states separated by a discontinuity which is usually located at the origin of the computational domain. This particular initial value problem is known as Riemann problem. The initial left and right uniform states are usually introduced by giving the density, the pressure and the velocity. The initial data for this problem is taken as \( (\rho_L, u_L, P_L) = (1, 0, 1) \) for the left state,
and, respectively \((\rho_R, u_R, P_R) = (0.125, 0, 0.1)\) for the right state. This initial set represents a tube where the left and the right regions are separated by a diaphragm, and filled by the same gas in two different physical states. If all the viscous effects are negligible along the tube walls and we are assuming that the tube is infinitely long to avoid reflections at the tube ends, solution of the Euler equations can be obtained using a wave analysis. Using a multi-step method to solve numerically problem (4) we can see in Fig. 3 the numerical approximation for density and also the filtered solution with a continuous line in the case of discretisation in which \(k = 3\). It is obvious that the filtered solution follows the numerical approximation accurately and the shocks are approximated in the right place. In all Figs. 3, 4, 5 we have fixed the mesh space at \(\Delta x = \frac{1}{20}\) in the physical domain \([-5, 5]\). We also use a Courant-Friederichs-Levy number of 0.475 needed for stability of numerical method based on Runge-Kutta of order four. The solution is calculated and shown at the final time \(T = 2.0\).

![Fig. 3 – Plot of density and filtered density at final time \(T = 2.0\).](image)

In Fig. 4 we present the numerical solution obtained for velocity \(u\) using the same discretisation with \(k = 3\). We can observe the overshooting and undershooting caused by the high-order method employed here. Again the filter solves this oscillations and replaces the numerical solution in the region of discontinuity with a correct left value \(u_L\) and a correct right value \(u_R\) in the precise place where the shocks form. In Fig. 5 we illustrate the plot of velocity and also the post-processed version of velocity. It is clear that a rarefaction wave appears as part of the solution and this rarefaction wave is approximated using a linear function between the left state and the right state.
In Fig. 5 we have presented a plot of pressure. For this test case the filtering procedure presented was based on a priori knowledge of the fact that the solution is formed by a rarefaction wave, a contact discontinuity and a simple shock. Following the same principles of the filtering we can filter out the numerical approximation making distinction between the type of shocks.

REFERENCES


