SOME FRACTIONAL GEOMETRICAL ASPECTS OF WEAK FIELD APPROXIMATION AND SCHWARZSCHILD SPACETIME

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Some interesting fractional geometrical aspects of weak field approximations, gravitational waves and Schwarzschild spacetime were discussed within the framework of fractional Riemann-Liouville action integral recently formulated by the author. We propose a natural method to obtain a Schwarzschild black hole embedded in the Friedmann-Robertson-Walker universe and to implement the cosmological constant “naturally” in Einstein field equations.

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I. INTRODUCTION: MOTIVATIONS

In recent years, growing attention has been focused on the importance of fractional derivatives and integrals in science. It is well believed today that fractional calculus is a quite irreplaceable means for description and investigation of classical and quantum complex dynamical system with holonomic as well as with nonholonomic constraints [1]. In simple words, the fractional derivatives and integrals describe more accurately the complex physical systems and at the same time, investigate more about simple dynamical systems. Dealing with fractional derivatives is not more complex than with usual differential operators. Its origin goes back to the 17th century, when in 1695 L’Hospital made some remarks to Leibniz about the mathematical meaning of fractional derivative of order 1/2. Leibniz’s response was an apparent paradox, from which one day useful consequences will be drawn. In these words fractional calculus was born and it was at the beginning a study reserved for the best minds in mathematics including J. Fourier, N. H. Abel, J. Liouville, B. Riemann,
H. Holmgren, etc., who contributed strongly to the fractional analysis program. However a serious research was first carried out by Liouville from 1832–1837, where he succeed to define the first fractional integration operator. Later on, further developments lead to the construction of the well-known Riemann-Liouville fractional integral operator playing a leading important role in complex dynamical systems. Since then the subject has evolved slowly and it is only in the last 20 years that the theory of fractional calculus has become more known particularly with the advent of the theory of Fractals. Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types, including Grunwald-Letnikov fractional derivative, Caputo fractional derivative, etc. but the Riemann-Liouville Operator is still the most frequently used when fractional integration is performed. In general, the continuous integro-differential operator is defined as

\[
a D_t^\alpha = \begin{cases} 
\frac{d^n}{dt^n} & \Re(\alpha) > 0, \\
1 & \Re(\alpha) = 0, \\
\int_a^t (t-\tau)^{-\alpha} d\tau & \Re(\alpha) < 0.
\end{cases}
\]

\(a\) and \(t\) are the limits of the integration and \(\Re(\alpha)\) denotes the real part of the fractional order \(\alpha\). Although the fractional theory is very rich, it was considered for more than three centuries as a theoretical mathematical field with no physical interests. But in reality, the pure mathematics has changed in some particular cases to meet the requirements of physical reality. In fact, most of the mathematical theories applicable to the study of fractional derivatives and integrals were developed prior to the turn of the 20th century, especially that numerous applications and physical manifestations of fractional calculus have been found. However, these applications and the mathematical background surrounding fractional calculus are far from paradoxical. While the physical meaning is difficult to grasp, the fractional definitions themselves are no more rigorous than those of their integer order counterparts. That is, the study fractional calculus opens new branches of thought and fills in the gaps of traditional standard calculus in ways that as of yet, no one completely assimilates or understands. But in reality, although various fields of application of fractional derivatives and integrals are already well done, some others have just started in particular the study of fractional problems of the Calculus of Variations (COV) and respective Euler-Lagrange type equations is a subject of current strong research and investigations. In 1996–97, F. Riewe used the COV with fractional derivatives and consequently obtained a version of the Euler-Lagrange equations (ELE) with fractional derivatives that combines the conservative and non-
conservative cases [2]. In 2001–2002, another approach was developed by M. Klimek by considering fractional problems of the COV but with symmetric fractional derivatives and correspondent ELE’s were obtained, using both Lagrangian and Hamiltonian formalisms [3]. In 2002, O. Agrawal extended Klimek problem and proved a formulation for variational problems with right and left fractional derivatives in the Riemann-Liouville sense [4]. In 2004 the ELE’s of Agrawal were used by D. Baling and T. Avkar to investigate problems with Lagrangians which are linear on the velocities [5]. In all the above mentioned studies, ELE’s depend on left and right fractional derivatives, even when the problem depend only on one type of them. In 2005, M. Klimek studied problems depending on symmetric derivatives for which ELE’s include only the derivatives that appear in the formulation of the problem [6]. The major problem with all these approaches is the presence of non-local fractional differential operators and the adjoint of a fractional differential operator used to describe the dynamics is not the negative of itself. Other complicated problems arise during the mathematical manipulations as the appearance of a very complicated Leibniz rule (the derivative of product of functions) and the non-presence of any fractional analogue of the chain rule. In general, the physical reasons for the appearance of fractional equations are long-range dissipation and nonconservation. For these reasons, it seems for us of interest to study the fractional Hamiltonian’s of nonconservative dynamical systems.

Recently, we proposed a novel approach known as fractional action-like variational approach (FALVA) or fractionally differentiated Lagrangian function (FDLF) to model nonconservative dynamical systems where fractional time integral introduces only one parameter $\alpha$ while in other models an arbitrary number of fractional parameters (orders of derivatives) appear [7–11]. The derived Euler-Lagrange equations are similar to the standard one but with the presence of fractional generalized external force acting on the system. No fractional derivatives appear in the derived equations. The conjugate momentum, the Hamiltonian and the Hamilton’s equations are shown to depend on the fractional order of integration $\alpha$ and vary as inverse of time.

In the present work, we will extend our formalism to Riemannian geometry where we will explore the role of fractional geodesic equation in weak field approximation, gravitational waves and Schwarzschild spacetime manifold. The paper is organized as follows: In section II, we review rapidly the basic concepts of FDLF on classical manifolds. In section III, we extend the FDLF formalism to differentiable manifolds where the fractional geodesic equation is obtained. We illustrate its physical importance by studying two examples: the fractional motion of relativistic particles (including photons) in the weakly curved spacetime metrics and the gravitational waves impinging on a spring. Some important consequences are discussed. In section IV, the role of fractional geodesic equations in modern mathematical astrophysics and cosmology, in
particular the Schwarzschild static spacetime is discussed. Conclusions are given in section V.

II. FRACTIONALLY DIFFERENTIATED LAGRANGIAN FUNCTION ON CLASSICAL MANIFOLD

In this section, we review briefly the basic mathematical formalism of FDLF or FALVA.

**Definition 1.** The Riemann-Liouville fractional integral is defined by:

$$
(t_0 = 0) \int_{t_0}^{t} f(\tau)(t-\tau)^{\alpha-1} d\tau, \quad 0 < \alpha < 1.
$$

(1)

**Definition 2 [7–10].** Consider a smooth Riemann manifold \( M \) and let \( L \) be an admissible smooth Lagrangian function \( L: R \times TM \rightarrow R \) on \( R \times R^d \times C^d \), \( d \geq 1 \). Given a Riemann-Liouville fractional function \( S_{0<\alpha<1} \) on the set of paths \( q(\tau), \quad 0 \leq \tau \leq t \) between two given points \( A = q(0) \) and \( B = q(t) \), i.e. a function on the tangent bundle \( TM \). For any piecewise smooth differentiable path \( q:[t_0,t_1] \rightarrow M \), the fractionally differentiated Lagrangian function (FDLF) associated \( S_{0<\alpha<1} \) to \( L \) is defined by

$$
S_{0<\alpha<1}[q] = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} L(\dot{q}(\tau), q(\tau), \tau)(t-\tau)^{\alpha-1} d\tau = 
$$

(2)

where \( L(\dot{q}, q, \tau) \) is the Lagrangian weighted with \( (t-\tau)^{\alpha-1}/\Gamma(\alpha) \) and \( \Gamma(1+\alpha)g_\mu(\tau) = t^\alpha - (t-\tau)^\alpha \) with the scaling properties \( g_{\mu(\tau)}(\tau) = \mu^\alpha g_\tau(\tau), \mu > 0 \).

**Theorem 1.** Let \( L: R \times TM \rightarrow R \) be a Lagrangian and \( x = x(\tau, \xi) \) be the coordinate point of \( q(\tau, \xi) \). Set \( L = L(x, \xi) \) evaluated at \( \xi = \dot{x} \) where \( x = (x^i) \) is a coordinate system on \( M \) and \( \xi^i = dx^i \) are functions on tangent vectors. Then the initial curves \( x = x(\tau) \) satisfies the fractional Euler-Lagrange equations (FEL)
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\[ \frac{\partial L}{\partial x^k} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^k} \right) = \frac{1 - \alpha}{t - \tau} \frac{\partial L}{\partial x^k} \equiv F_{\dot{x}^i}^{\alpha}, \quad (3) \]

### III. FRACTIONALLY DIFFERENTIATED LAGRANGIAN FUNCTION ON DIFFERENTIABLE MANIFOLD

One can extend the previous arguments to differentiable manifold by stating the following corollary.

**Corollary 1.** Fractional geodesic motion corresponds to constant velocity and it gives an extremum of the fractional action (2) with \( L = \left( \frac{1}{2} \right) m g_{ij} \dot{q}^i \dot{q}^j \), where \( g_{ij} \) is the metric, i.e. an arbitrary function of the coordinates, \( \dot{q} = dq/d\tau = u \) and \( m \) is the particle mass assumed equal to one for simplicity.

The equation of motion is then given by

\[ \frac{d^2 q^j}{d\tau^2} + \frac{\alpha - 1}{\tau - t} \frac{d q^j}{d\tau} + \Gamma^j_{ij} \frac{d q^i}{d\tau} = 0. \quad (4) \]

**Proof.** Given \( L = \left( \frac{1}{2} \right) g_{ij} \dot{q}^i \dot{q}^j \). The fractional Euler-Lagrange equations

\[ \frac{\partial L}{\partial q^k} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}^k} \right) = \frac{1 - \alpha}{t - \tau} \frac{\partial L}{\partial q^k}, \quad (5) \]

give

\[ \frac{1}{2} \ddot{q}^j \frac{\partial g_{ij}}{\partial q^k} - \frac{d}{d\tau} \left( \frac{1}{2} \left[ g_{ij} \dot{q}^i + g_{ik} \dot{q}^k \right] \right) = \frac{1 - \alpha}{t - \tau} \left( \frac{1}{2} \left[ g_{ij} \dot{q}^i + g_{ik} \dot{q}^k \right] \right). \]

Cancel the halves, expand the time derivative, use the chain rule and the symmetry of \( g \), we find:

\[ g_{kj} \ddot{q}^j + g_{kj} \frac{\alpha - 1}{\tau - t} \dot{q}^j + \frac{1}{2} \left( \frac{\partial g_{kj}}{\partial q^i} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^k} \right) \dot{q}^i \dot{q}^j = 0, \]

or in the following form

\[ g_{kj} \ddot{q}^j + g_{kj} \frac{\alpha - 1}{\tau - t} \dot{q}^j + \left[ ij, k \right] \dot{q}^i \dot{q}^j = 0, \]

where

\[ \left[ ij, k \right] = \frac{1}{2} \left( \frac{\partial g_{kj}}{\partial q^i} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^k} \right), \]

is the Christoffel symbol of first kind. Multiplying throughout by the inverse \( g^{ik} \) and arranging gives the required result:
\[
\frac{d^2 q^l}{d\tau^2} + \frac{\alpha - 1}{\tau - t} \frac{dq^l}{d\tau} + \Gamma^l_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau} = 0,
\]

where \( \Gamma^l_{ij} = g^{lk} \{ij, k\} \) is the Christoffel symbol of second kind.

**Lemma 1.** Equations (4) can be written as a system of first order equation with integral curves:

\[
\frac{dq^l}{d\tau} = v^l, 
\]

\[
\frac{dv^l}{d\tau} + \frac{\alpha - 1}{\tau - t} v^l + \Gamma^l_{ij} v^i v^j = 0. 
\]

In fact, equation (7) can be written in fact as:

\[
\frac{dv^l}{d\tau} = -\Gamma^l_{ij} v^i v^j + \frac{1 - \alpha}{\tau - t} v^l \equiv -\Gamma^l_{ij} v^i v^j + F^l,
\]

where

\[
F^l = \frac{1 - \alpha}{\tau - t} v^l,
\]

is the decaying forcing term. From a control theory point of view, \( F^l \) is the corresponding input weak decaying vector field. In fact, by defining \( \dot{q}^\alpha = dq^\alpha / dT = Q^\alpha \), equation (8) is identical to a Langevin equation with a time-dependent friction term in case a random source characterizing the properties of medium where motion occurs is applied (for example a random or stochastic electromagnetic field) [12].

**Corollary 2.** If the metric does not depend on one of the coordinate \( x^a \), i.e. the time, or an angle, then the canonical momentum for the Lagrangian \( L = (1/2) m g_{\mu\nu} u^\mu u^\nu \) is not conserved.

**Proof.** We can show this explicitly: the fractional geodesic equation can be written consequently as:

\[
m \frac{dp^\beta}{d\tau} + m \frac{\alpha - 1}{\tau - t} p^\beta - \Gamma^\gamma_{\beta\delta} p^\delta p^\gamma = 0.
\]

Using the fact:

\[
\Gamma^\gamma_{\beta\delta} p^\delta p^\gamma - \frac{1}{2} \left( g_{\nu\beta,\gamma} + g_{\nu\gamma,\beta} - g_{\beta\delta,\nu} \right) p^\delta p^\gamma,
\]

and that two terms cancel due to the symmetry of \( \delta \) and \( \nu \), we have finally:

\[
m \frac{dp^\beta}{d\tau} + m \frac{\alpha - 1}{\tau - t} p^\beta = \frac{1}{2} g_{\nu\delta,\beta} p^\nu p^\delta.
\]
making the nonconservation obvious. We have used the simplified notation
\[ g_{ij,k} = \partial g_{ij} / \partial q^k. \] Conservation occurs when \( \alpha = 1. \)

We illustrate the importance of the FDLF on differentiable manifold by the
following two examples:

a) First note that one can consider the fractional motion of relativistic
particles (including photons) in the weakly curved spacetime metrics defined in
general by
\[ ds^2 = (1 + 2\Phi) d\tau^2 - (1 - 2\Phi) (dx^2 + dy^2 + dz^2) \]
where \( \Phi \) is the gravitational potential [13]. But now, we are interested in nonrelativistic
particles. The fractional geodesic equation in lower components reads:
\[ \frac{dp_0}{d\tau} + m \frac{\alpha - 1}{\tau - t} p_0 = \frac{1}{2} g_{\nu\delta,\beta} P^\nu P^\delta. \] (10)

In the nonrelativistic limit, \( p_0^0 = m. \) Consequently, \( g_{\nu\delta,\beta} P^\nu P^\delta \approx 2m^2\Phi_{,\mu}, \)
where \( \Phi_{,\mu} \equiv \partial \Phi / \partial q^\mu. \) The fractional geodesic equation then reduces to
\[ \frac{dp_0}{d\tau} + m \frac{\alpha - 1}{\tau - t} p_0 = m \frac{\partial \Phi}{\partial \tau}. \] (11)
\[ \frac{dp^i}{d\tau} + m \frac{\alpha - 1}{\tau - t} p^i = -m \nabla_i \Phi. \] (12)

Equation (11) deals with the nonconservation of energy even for static
gravitational field and equation (12) corresponds to the fractional Newton’s
second law. In fact, the fractional geodesic equations or the fractional Euler-
Lagrange equations could have additional interesting consequences in weakly
curved spacetimes. If special relativity is nearly correct, so that \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \)
where \( \eta_{\mu\nu} = \text{diag}(1,-1,-1,-1) \) in the usual special relativity Minkowskian
metric and \( |h_{\mu\nu}| << 1 \) is a small correction. In vacuum, the fractional linearized
equations in the Lorentz gauge become [7–10]:
\[ \left\{ \left( \frac{\partial^2}{\partial \tau^2} + \frac{\alpha - 1}{\tau - t} \frac{\partial}{\partial \tau} \right) - \left( \nabla^2 + \frac{\alpha - 1}{\chi^4} \nabla \right) \right\} h^{\beta\delta} = 0, \] (13)
where \( \nabla^2 \equiv \nabla \nabla = \partial^2 / \partial \tau^2 \). The solution is given by
\[ \bar{h}^{\beta\delta}(\xi) = A^{\beta\delta} \left( m \sqrt{\xi} \right)^{1-n/2} Z_{1-n/2}(im \sqrt{\xi}), \]
where \( \xi = (T-x)(T+x), \) \( n = 2\alpha \) and \( T = t - \tau. \) Here \( h_{\beta\delta} \equiv \bar{h}_{\beta\delta} - (1/2)\eta_{\beta\delta} \bar{h}, \)
\( \bar{h}_{\beta\delta} = h_{\beta\delta} - (1/2)\eta_{\beta\delta} h \) and \( A^{\beta\delta} \) is some vector. Using the Lorentz gauge condition
\( \bar{h}_{\beta} \equiv 0, \) the fractional geodesic equations for a free particle initially at rest are:
so that stationary particles are not at rest. In contrary, we have \( u^\beta \propto (\tau - t)^{1-\alpha} \), \( 0 < \alpha < 1 \). In other words, the velocity is accelerated with time and consequently \( q^\beta \propto (\tau - t)^{2-\alpha} \), i.e. the proper distance between two particles will increase rapidly with time. The masses are rapidly moving and on expect that there exists a direct friction force acting on the masses due to the gravitational waves.

b) If one considers a gravitational wave impinging on a spring in the \( x \) direction, with mass \( m \), natural length \( l_0 \), damping \( \nu \) and spring constant \( k \) and with fractional proper length given by [14]:

\[
l_{0<\alpha<1}(\tau) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \sqrt{1 + h_{xx}(\tau)(t - \tau)^{\alpha-1}} \, d\tau,
\]

where \( h_{xx} \) is the transverse traceless gauge, then ignoring higher order terms, one can easily deduce the fractional sinusoidal forcing to the spring as:

\[
\dot{\xi} + \left( \frac{\alpha-1}{\tau-t} + 2\gamma \right) \dot{\xi} + \omega_0^2 \xi = \frac{1}{2} l_0 h_{xx},
\]

where \( \xi = l - l_0 \), \( \gamma = \nu/m \) and \( \omega_0^2 = 2k/m \). In order to detect gravitational waves, one should adjust the natural frequency as close as possible to the expected frequency of the gravitational waves, and makes the damping decreases linearly with time as \( \gamma = (1-\alpha)/2(\tau-t) \).

IV. THE ROLE OF FRACTIONAL GEODESIC EQUATIONS IN MODERN MATHEMATICAL ASTROPHTICS AND COSMOLOGY

In return to differential geometry on manifolds, we would like to show the important role of fractional geodesic equations in modern mathematical astrophysics and cosmology, in particular in this paper, the Schwarzschild static spacetime.

**Lemma 2.** Let \( R_{ab} = g^{ij} R_{abij} \) be the Ricci tensor where

\[
R_{abcd} = \frac{1}{2} \left( g_{bc,ad} - g_{bd,ac} + g_{ad,bc} - g_{ac,bd} \right) + \Gamma_{ad}^j \Gamma_{bjc} - \Gamma_{ac}^j \Gamma_{bjd},
\]

is the covariant curvature tensor; \( \Gamma_{ij}^k = g_{pl} \Gamma_{ik}^p \). Then the time-component \((00)\) of the Ricci scalar curvature in the weak field approximation is \( R_{00} \approx \partial_t \Gamma_{00} = -(1/2) \partial^2 h_{00} \) [13, 14].
**Corollary 3.** The fractional geodesic equations reduce to:
\[ \text{div} \gamma^i + \frac{\alpha - 1}{\tau} \partial^i v^i + c^2 R_{00} = 0, \] (18)

\( c \) being the celerity of light (in natural unit), \( v^i = dx^i/d\tau \) the particle velocity and \( \text{div} \gamma^i = \partial^i \gamma^i = -4\pi G \rho \) being the Poisson equation, \( \rho = 3M/4\pi R^3 \) the mass density in gravitational interaction, \( G \) is Newton gravitational constant and \( \gamma \) the acceleration. It follows that \( R_{00} = 4\pi \rho G_{\text{eff}} / c^2 \) where
\[ G_{\text{eff}} = G \left( 1 + \frac{1 - \alpha}{4\pi \rho G T} \partial^i v^i \right) = G + \Delta G, \] (19)
is the effective Newton gravitational constant.

**Theorem 2.** If a fractional decaying friction force represented by equation (9) is added to the standard geodesic equation, the Newton gravitational constant is perturbed by \( \Delta G = \left( (1 - \alpha) / 4\pi \rho T \right) \partial^i v^i. \)

In what follows, we will be interested on the vacuum Einstein field equations \( R_{ab} = 0. \) A well-know solution describing the gravitational field outside a spherically symmetric body at rest corresponds to the Schwarzschild metric. We will show that when the gravity is fractionally perturbed by \( \Delta G, \) we will fall into a Schwarzschild black hole embedded in the black ground of Friedman-Robertson-Walker (FRW) expanding universe.

**Definition 3.** The Raychaudhuri expansion scalar factor \( \dot{\theta} \) is defined by [13]:
\[ \partial_i v^i = 3\dot{\theta} = 3 \dot{a} / a. \] (20)

\( a = a(\tau) \) is the scale factor of the universe.

**Corollary 4.** The perturbed gravity reduces to
\[ \Delta G = \frac{3(1 - \alpha) \dot{a}}{4\pi \rho T a}. \] (21)

**Corollary 5.** For \( r \leq R, \ \rho = 3m/4\pi r^3, \) the fractional perturbed Schwarzschild metric is given by:
\[ ds^2 = -c^2 \left( 1 - \frac{2 MG}{rc^2} - \frac{2\beta(1-\alpha)r^2H^2}{c^2} \right) dt^2 + \right. \] (22)
\[ + \left. \left( 1 - \frac{2 MG}{rc^2} - \frac{2\beta(1-\alpha)r^2H^2}{c^2} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]
where \( H = \dot{a} / a = 1/(\beta T) \) is the Hubble parameter, \( \beta \) is a positive parameter.
This is the required result: a Schwarzschild black hole embedded in the black ground of Friedman-Robertson-Walker (FRW) expanding universe. When $H = 0$ or $\alpha = 1$, it is just the standard Schwarzschild metric and when $M = 0$, it is the de-Sitter metric. Note that $\beta(1 - \alpha) > 0$, and consequently anti-de-Sitter spacetime are forbidden in our fractional dynamics. The term $6\beta(1 - \alpha)H^2/c^2$ could be also interpreted as a cosmological constant. Consequently, equation (22) takes the form:

$$ds^2 = -c^2\left(1 - \frac{2MG}{rc^2} - \frac{\Lambda_{\alpha}r^2}{3c^2}\right)dt^2 + \left(1 - \frac{2MG}{rc^2} - \frac{\Lambda_{\alpha}r^2}{3c^2}\right)^{-1}dr^2 +$$

$$+r^2\left(d\theta^2 + \sin^2 \theta d\phi^2\right),$$

where

$$\Lambda_{\alpha} = \frac{6\beta(1 - \alpha)H^2}{c^2},$$

is identified to the fractional cosmological constant.

**Proposition.** There is no need to implement the cosmological constant in Einstein field equations; all what we need to do is to start from a fractional functional action integral and derive the fractional geodesic equations. A weak decaying force which looks like equation (9) will appear, perturbing in the case of weak field approximation the gravitational constant and consequently, the perturbed part is identified naturally to a cosmological Einstein lambda.

V. CONCLUSIONS

It is important to note that a new way to implement the cosmological constant or Einstein lambda type is at work in our fractional scenario. Despite the important question that arises “what does the frictional force mean, in the vacuum?” we showed that the FDLC and consequently the fractional geodesic equation and the resulted perturbed gravity lead to various modification to the standard prediction of general relativity. There is nothing in the current notion of physical space that entails the presence of the decaying friction term. Some others consequences, astrophysical, cosmological applications and fractional features are under progress.

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