A NEW OPTIMAL BOUND ON LOGARITHMIC SLOPE OF ELASTIC HADRON-HADRON SCATTERING

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In this paper we prove a new optimal bound on the logarithmic slope of the elastic slope \( b \) when: \( \sigma_{el} \) and \( \frac{d\sigma}{d\Omega}(1) \) and \( \frac{d\sigma}{d\Omega}(-1) \), are known from experimental data. The results on the experimental tests of this new optimal bound are presented in Sect. 3 for the principal meson-nucleon elastic scatterings: \((\pi\pm P \rightarrow \pi\pm P \) and \( K \pm P \rightarrow K \pm P)\) at all available energies. Then we show that the saturation of this optimal bound is observed with high accuracy practically at all available energies in meson-nucleon scattering.

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1. INTRODUCTION

Recently, in Ref. [1], by using reproducing kernel Hilbert space (RKHS) methods [2–4], we described the quantum scattering of the spinless particles by a principle of minimum distance in the space of the scattering quantum states (PMD-SQS). Some preliminary experimental tests of the PMD-SQS, even in the crude form [1], when the complications due to the particle spins are neglected, showed that the actual experimental data for the differential cross sections of all \( PP, \bar{P}P, K \pm P, \pi \pm P \), scatterings at all energies higher than 2 GeV, can be well systematized by PMD-SQS predictions. Moreover, connections between the optimal states [1], the PMD-SQS in the space of quantum states and the maximum entropy principle for the statistics of the scattering channels was also recently established by introducing quantum scattering entropies [5–8].

The aim of this paper is to prove a new optimal bound on the logarithmic slope of the elastic hadron-hadron scattering by solving the following optimization problem: to find an lower bound on the logarithmic slope \( b \) when: \( \sigma_{el}, \frac{d\sigma}{d\Omega}(+1) \) and \( \frac{d\sigma}{d\Omega}(-1) \), including spin effects, are given. The results on the experimental tests of this new optimal bound are presented for the principal meson-nucleon elastic scatterings: \((\pi \pm P \rightarrow \pi \pm P \) and \( K \pm P \rightarrow K \pm P)\) at all available energies.
available energies. Then it was shown that the saturation of this optimal bound is observed with high accuracy practically at all available energies in meson-nucleon scattering.

2. OPTIMAL HELICITY AMPLITUDES FOR SPIN \((0\rightarrow 1/2^+ \rightarrow 0\rightarrow 1/2^+)\) SCATTERINGS

First we present some basic definitions and results for the optimal states in the meson-nucleon scattering when the integrated elastic cross section \(\sigma_{el}\) and differential cross sections \(\frac{d\sigma}{d\Omega}(\pm 1)\) are known from experiments. Therefore, let \(f_{++}(x)\) and \(f_{+-}(x)\), \(x \in [-1, 1]\), be the scattering helicity amplitudes of the meson-nucleon scattering process:

\[
M(0^-) + N(1/2^+) \rightarrow M(0^-) + N(1/2^+) \tag{1}
\]

\(x = \cos \theta\), \(\theta\) being the c.m. scattering angle. The formalizations of the helicity amplitudes \(f_{+}(x)\) and \(f_{-}(x)\) are chosen such that the differential cross section \(\frac{d\sigma}{d\Omega}(x)\) is given by

\[
\frac{d\sigma}{d\Omega}(x) = |f_{++}(x)|^2 + |f_{+-}(x)|^2 \tag{2}
\]

Then, the elastic integrated cross section \(\sigma_{el}\) is given by

\[
\frac{\sigma_{el}}{2\pi} = \int_{-1}^{+1} d\frac{\sigma}{d\Omega}(x) dx = \int_{-1}^{+1} \left[ |f_{++}(x)|^2 + |f_{+-}(x)|^2 \right] dx \tag{3}
\]

Since we will work at fixed energy, the dependence of \(\sigma_{el}\) and \(\frac{d\sigma}{d\Omega}(x)\) and of \(f(x)\), on this variable was suppressed. Hence, the helicities of incoming and outgoing nucleons are denoted by \(\mu\), \(\mu'\), and was written as \((+), (-)\), corresponding to \((\frac{1}{2})\) and \((-\frac{1}{2})\), respectively. In terms of the partial waves amplitudes \(f_{J^+}\) and \(f_{J^-}\) we have

\[
\left\{
\begin{aligned}
f_{++}(x) &= \sum_{J=\frac{1}{2}}^{J_{\text{max}}} \left( J + \frac{1}{2} \right) (f_{J^-} + f_{J^+}) d_{11}^J(x) \\
f_{+-}(x) &= \sum_{J=\frac{1}{2}}^{J_{\text{max}}} \left( J + \frac{1}{2} \right) (f_{J^-} - f_{J^+}) d_{11}^J(x)
\end{aligned}
\right. \tag{4}
\]

where the \(d_{\mu\nu}^J(x)\)-rotation functions are given by

\[
\left\{
\begin{aligned}
d_{11}^J(x) &= \frac{1}{l+1} \left[ \frac{1+x}{2} \right]^\frac{l}{2} \left[ P_{l+1}(x) - P_l(x) \right] \\
d_{11}^J(x) &= \frac{1}{l+1} \left[ \frac{1-x}{2} \right]^\frac{l}{2} \left[ P_{l+1}(x) + P_l(x) \right]
\end{aligned}
\right. \tag{5}
\]
and prime indicates differentiation of Legendre polynomials $P_j(x)$ with respect to $x = \cos \theta$.

$$\frac{\sigma_{el}}{2\pi} = \sum (2J + 1) \left[ |f_{j+}|^2 + |f_{j-}|^2 \right]$$  \hspace{1cm} (6)

Now, let us consider the optimization problem

$$\left\{ \min \left[ \sum (2J + 1) \left( |f_{j+}|^2 + |f_{j-}|^2 \right) \right] \right. \text{subject to:}$$

$$\frac{d\sigma}{d\Omega} (+1) = \text{fixed, } \text{and} \frac{d\sigma}{d\Omega} (-1) = \text{fixed}$$

which will be solved by using Lagrange multiplier method [9] where

$$\begin{align*}
\mathcal{L} &= \left[ \sum (2J + 1) \left( |f_{j+}|^2 + |f_{j-}|^2 \right) \right] + \\
&+ \alpha \left[ \frac{d\sigma}{d\Omega} (+1) - \sum (J + 1/2) (f_{j+} + f_{j-})^2 \right] \\
&+ \beta \left[ \frac{d\sigma}{d\Omega} (-1) - \sum (J + 1/2) (f_{j+} - f_{j-})^2 \right]
\end{align*}$$  \hspace{1cm} (8)

So, we prove that the solution of the problem (7)–(8) is as follows

$$\begin{cases}
f_{++}^o(x) = f_{++}(+1) - \frac{K_{\frac{1}{2}\frac{1}{2}}(x,+1)}{K_{\frac{1}{2}\frac{1}{2}}(+1,+1)} \\
f_{+-}^o(x) = f_{+-}(-1) - \frac{K_{\frac{1}{2}\frac{1}{2}}(x,-1)}{K_{\frac{1}{2}\frac{1}{2}}(-1,-1)}
\end{cases}$$  \hspace{1cm} (9)

where the reproducing kernel functions are defined as

$$\begin{align*}
K_{\frac{1}{2}\frac{1}{2}}(x,y) &= \sum_{J} J \left( J + \frac{1}{2} \right) d_{\frac{1}{2}}^J (x) d_{\frac{1}{2}}^J (y) \\
2K_{\frac{1}{2}\frac{1}{2}}(+1,+1) &= (J_o + 1)^2 - 1/4 \\
2K_{\frac{1}{2}\frac{1}{2}}(-1,-1) &= (J_o + 1)^2 - 1/4 \\
(J_o + 1)^2 - \frac{1}{4} &= \frac{4\pi}{\sigma_{el}} \left[ \frac{d\sigma}{d\Omega} (1) + \frac{d\sigma}{d\Omega} (-1) \right]
\end{align*}$$  \hspace{1cm} (10)
Proof: Let us consider the complex partial amplitudes \( f_{J^z} = r_{J^z} + ia_{J^z} \), where \( r_{J^z} \) and \( a_{J^z} \) are real and imaginary parts, respectively. Then, Eq. (8) can be expressed completely in terms of the variational variables \( r_{J^z} \) and \( a_{J^z} \). Therefore, by calculating the first derivative we obtain

\[
\begin{align*}
\frac{1}{(2J+1)} \frac{\partial \mathcal{L}}{\partial r_{J^z}} &= r_{J^z} - \alpha R^{++}(+1) \pm \beta R^{--}(-1) = 0 \\
\frac{1}{(2J+1)} \frac{\partial \mathcal{L}}{\partial a_{J^z}} &= a_{J^z} - \alpha A^{++}(+1) \pm \beta A^{--}(-1) = 0
\end{align*}
\] (11)

where we have defined \( f^{++}(x) \equiv R^{++}(x) + iA^{++}(x) \), and \( f^{--}(x) \equiv R^{--}(x) + iA^{--}(x) \), respectively, where

\[
\begin{align*}
R^{++}(+1) &= \sum \left(J + \frac{1}{2}\right)(r_{J^z} + r_{J^-}) \\
A^{++}(+1) &= \sum \left(J + \frac{1}{2}\right)(a_{J^z} + a_{J^-}) \\
R^{--}(-1) &= \sum \left(J + \frac{1}{2}\right)(r_{J^-} - r_{J^z}) \\
A^{--}(-1) &= \sum \left(J + \frac{1}{2}\right)(a_{J^-} - a_{J^z})
\end{align*}
\] (12)

Therefore, from Eqs (11) we get

\[
\begin{align*}
\left\{ \begin{array}{l}
r_{J^z} = \alpha R^{++}(+1) - \beta R^{--}(-1) \\
r_{J^-} = \alpha R^{++}(+1) + \beta R^{--}(-1) \\
a_{J^z} = \alpha A^{++}(+1) - \beta A^{--}(-1) \\
a_{J^-} = \alpha A^{++}(+1) + \beta A^{--}(-1)
\end{array} \right.
\] (13)

Then using the definitions (2) and (3), we get

\[
\alpha^{-1} = \beta^{-1} = (J_o + 1)^2 - 1/4 = \frac{4\pi}{\sigma_{el}} \left[ \frac{d\sigma}{d\Omega}(+1) + \frac{d\sigma}{d\Omega}(-1) \right]
\] (14)

and, consequently we obtain that the optimal solution of the problem (7) can be written in the form

\[
\begin{align*}
f_{++}^o(x) &= \frac{2f_{++}(+1)}{(J_o + 1)^2 - 1/4} \sum_{J_{1/2}} \left(J + \frac{1}{2}\right) d_{11}^{J^z}(x) d_{11}^{J^z}(+1) \\
f_{--}^o(x) &= \frac{2f_{++}(-1)}{(J_o + 1)^2 - 1/4} \sum_{J_{1/2}} \left(J + \frac{1}{2}\right) d_{11}^{J^z}(x) d_{11}^{J^z}(-1)
\end{align*}
\] (15)
Now from Eqs. (14) and (15) we obtain the optimal solution (9) in which the reproducing functions \( K_{\frac{1}{2}}(x, y) \) and \( K_{\frac{3}{2}}(x, y) \) are defined by (10).

### 3. OPTIMAL BOUND ON LOGARITHMIC SLOPE

We recall the definition of the elastic slope \( b \), and the relation

\[
\sigma_{\text{el}} \left( \frac{d}{d\Omega} (1) \right) \sigma_{\Omega} \lambda = \left[ \ln \left( \frac{d\sigma}{d\Omega} (x) \right) \right]_{x=1} \quad (16)
\]

where transfer momentum is defined by: \( t = -2q^2(1-x) \), \( \overline{\lambda} = 1/q \), and \( q \) is the c.m momentum.

Now, let us assume that \( \sigma_{\text{el}}, (1) \frac{d}{d\Omega} (1) \sigma_{\Omega} \) and \( (1) \frac{d}{d\Omega} (1) \sigma_{\Omega} \) are known from the experimental data. Then, taking into account the solution (9)–(10) of the optimization problem (7), it is easy to prove that the elastic slope \( b \) defined by (16) must obey the optimal inequality:

\[
b \geq b_o = \overline{\lambda}^2 \left\{ \frac{4\pi}{\sigma_{\text{el}}} \left[ \frac{d\sigma}{d\Omega} (1) + \frac{d\sigma}{d\Omega} (1) \right] - 1 \right\} \quad (17)
\]

**Proof:** Indeed a proof of the optimal inequality (17) can be obtained as singular solution of the following optimization problem

\[
\min \{b\}, \text{ subject to: } \sigma_{\text{el}} = \text{fixed}, \quad \frac{d\sigma}{d\Omega} (1) = \text{fixed}, \quad \frac{d\sigma}{d\Omega} (1) = \text{fixed} \quad (18)
\]

So, the lower limit of the elastic slope \( b \) is just the elastic of the differential cross section given by the result (9)–(10). Consequently, we obtain that the optimal slope \( b_o \) is given by

\[
b_o = \overline{\lambda}^2 \left[ \frac{K_{\frac{1}{2}}(x, +1)}{K_{\frac{3}{2}}(x, +1)} \right]_{x=1} = \overline{\lambda}^2 \left[ \left( J_o (J_o + 2) - \frac{1}{4} \right) \right] \quad (19)
\]

Then, using the second part of (14) we obtain the inequality (17).

An important model independent result obtained Ref. [1], via the description of quantum scattering by the principle of minimum distance in space of states (PMD-SS), is the following optimal lower bound on logarithmic slope of the forward diffraction peak in hadron-hadron elastic scattering:
\[
 b \geq b_o \geq \frac{\pi^2}{4} \left[ \frac{4\pi}{\sigma_{el}} \frac{d\sigma}{d\Omega}(1)-1 \right]
\]  

(20)

In is important to remark, the optimal bound (17) improves in a more general and exact form not only the unitarity bounds derived by MacDowell and Martin [10] for the logarithmic slope \( b_A \) of absorptive contribution \( \frac{d\sigma_A}{d\Omega}(s,t) \) to the elastic differential cross sections but also the unitarity lower bound derived in Ref. [1] (see also Ref. [11, 15]) for the slope \( b \) of the entire \( \frac{d\sigma}{d\Omega}(s,t) \) differential cross section. Therefore, it would be important to make an experimental detailed investigation of the saturation of this bond in the hadron-hadron scattering, especially in the low energy region.

4. EXPERIMENTAL TESTS OF THE BOUND (17)

A comparison of the experimental elastic slopes \( b \) with the optimal slope \( b_o \), (17) is presented in Figs. 1 for \((\pi^\pm P, K^\pm P)\)-scatterings: The values of the \( \chi^2 = \sum_j (b_j - b_{oj})^2 / (\epsilon_{b,j}^2 + \epsilon_{b_{oj},j}^2) \), (where \( \epsilon_{b,j} \) and \( \epsilon_{b_{oj},j} \) are the experimental errors corresponding to \( b \) and \( b_{oj} \), respectively) are used for the estimation of departure from the optimal PMD-SS-slope \( b_{oj} \), and then, we obtain the statistical parameters presented in Table 1. For \( \pi^\pm P \)-scattering the experimental data on \( b, \frac{d\sigma}{d\Omega}(1), \), \( \frac{d\sigma}{d\Omega}(-1), \) and \( \sigma_{el} \), for the laboratory momenta in the interval \( 0.2 \text{GeV} \leq P_{LAB} \leq 10 \text{GeV} \) are calculated directly from the phase shifts analysis (PSA) of Hohler et al. [12]. To these data we added some values of \( b \) from the linear fit of Lasinski et al. [14] and also from the original fit of authors quoted in some references in [15]. Unfortunately, the values of \( b_{oj} \) corresponding to the Lasinski’s data [14] was impossible to be calculated since the values of \( \frac{d\sigma}{d\Omega}(1) \) from their original fit are not given. For \( K^\pm P \)-scatterings the experimental data on \( b, \frac{d\sigma}{d\Omega}(1), \frac{d\sigma}{d\Omega}(-1) \) and \( \sigma_{el} \), in the case of \( K^\mp P \), are calculated from the experimental (PSA) solutions of Arndt et al. [13]. To these data we added those collected from the original fit of data from references of [15] which the approximation \( \frac{d\sigma}{d\Omega}(-1)=0 \). For \( K^+ P \)-scattering, we added some values of \( b \) from the linear fit of Lasinski et al. [14] and also those pairs \( (b, b_{oj}) \) calculated directly from the experimental (PSA) solutions of Arndt et al. [13]. All these results can be compared with those presented in [15].
5. SUMMARY AND CONCLUSION

The main results and conclusions obtained in this paper can be summarized as follows:

(i) In this paper we proved the optimal bound (17) as the singular solution ($\lambda_0 = 0$) of the optimization problem to find a lower bound on the logarithmic slope $b$ with the constraints imposed when $\sigma_{el}$ and $\frac{d\sigma}{d\Omega}(+1)$ and $\frac{d\sigma}{d\Omega}(-1)$ are fixed from experimental data. This result is similar with that obtained recently in Refs. [1, 15] for the problem to find an upper bound for the scattering entropies when $\sigma_{el}$ and $\frac{d\sigma}{d\Omega}(+1)$ are fixed.

(ii) We find that the optimal bound (17) is verified experimentally with high accuracy at all available energies for all the principal meson-nucleon scatterings.

(iii) From mathematical point of view, the PMD-SQS-optimal states (9)–(10), are functions of minimum constrained norm and consequently can be completely described by reproducing kernel functions (see also Ref. [1, 3–4]). So, with this respect the PMD-SQS-optimal states from the reproducing kernel Hilbert space (RKHS) of the scattering amplitudes are analogous to the coherent states from the RKHS of the wave functions.

(iv) The PMD-SQS-optimal state (9)–(10) have not only the property that is the most forward-peaked quantum state but also possesses many other peculiar properties such as maximum Tsallis-like entropies, as well as the scaling and the s-channel helicity conservation properties, etc., that make it a good candidate for the description of the quantum scattering via an optimum principle. In fact the validity of the principle of least distance in space of states in hadron-hadron scattering is already well illustrated in Fig. 1 and Table 1.

All these important properties of the optimal helicity amplitudes (9)–(10) will be discussed in more detail in a forthcoming paper.

REFERENCES

Fig. 1 – The experimental values (black circles) of the logarithmic slope $b$ for the principal meson-nucleon scatterings are compared with the optimal PMD-SQS-predictions $b_o$ (white circles). The experimental data for $b$, $\frac{d\sigma}{d\Omega} (+1)$ and $\sigma_{\text{el}}$ are taken from Refs. [12–14], (see the text).
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Table 1

\( \chi^2 \) – statistical parameters of the principal hadron-hadron scattering. In these estimations for \( P_{\text{LAB}} \leq 2 \text{ GeV/c} \) the errors \( \epsilon_i^{\text{PSA}}(\pi^0 P) = 0.1 b^{\text{PSA}} \) and \( \epsilon_i^{\text{PSA}}(K^0 P) = 0.1 b^{\text{PSA}} \) are taken into account while for the errors to the optional slopes \( b_o \) calculated from phase shifts analysis and [15]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Statistical parameters} & \text{For} P_{\text{LAB}} \geq 2 \text{ GeV/c} & \text{For all} P_{\text{LAB}} \geq 0.2 \text{ GeV/c} \\
\hline
\pi^+ P \rightarrow \pi^+ P & N_p & \chi^2/n_{\text{ dof}} & N_p & \chi^2/n_{\text{ dof}} \\
\pi^- P \rightarrow \pi^- P & 28 & 1.02 & 90 & 3.37 \\
K^+ P \rightarrow K^+ P & 31 & 0.92 & 93 & 8.00 \\
K^- P \rightarrow K^- P & 37 & 1.15 & 73 & 1.91 \\
PP \rightarrow PP & 37 & 1.52 & 73 & 7.84 \\
\bar{PP} \rightarrow \bar{PP} & 29 & 5.01 & 32 & 5.06 \\
\hline
\end{array}
\]