A NEW CLASS OF HIGHER DIMENSIONAL COSMOLOGICAL MODELS OF UNIVERSE WITH VARIABLE G AND \( \Lambda \)-TERMS

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A new class of exact solutions of Einstein’s field equations with a perfect fluid source, variable gravitational coupling \( G \) and cosmological term \( \Lambda \) for FRW space-time in higher dimensions is obtained. The nature of the variables \( G(t) \), \( \Lambda(t) \), and the energy density \( \rho(t) \) have been examined for three cases. It is shown that in these models particle horizon exist and the cosmological term is decaying with time. Further, it is observed that new class of solutions include some previous cases in four-dimensional space-time. The results of the present studies are well within the range of observational limits.

Key words: cosmology, higher dimensions, variable gravitational and cosmological terms.

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1. INTRODUCTION

The exact physical situation at very early stages of the formation of our universe provoked great interest among researchers. Higher dimensional space-time has taken considerable research interest in an attempt to unify gravity with other forces in nature. This idea is particularly important in the field of cosmology since one knows that our universe was much smaller in its early stage than it is today. In this connection a number of attempts have been made to study the role of gravity with other fundamental forces in nature. The most famous five dimensional theory proposed by Kaluza [1] and Klein [2], was the first theory in which gravitation and electromagnetism could be unified in a single geometrical structure. The Kaluza-Klein’s idea to consider the coefficient of the fifth
co-ordinate as constant, was generalized by Thiry [3] and Jordan [4]. Marciano [5] has suggested that the experimental detections of time variations of the fundamental constants could be strong evidence of extra dimensions. To achieve unification of all interactions including weak and strong forces, many authors [6, 7] have extended the Kaluza-Klein formalism to higher dimensions. The investigations of super-string theory and super gravitational theory have created renewed interest among theoretical physicists to study the physics in higher dimensional space-time [8, 9]. Multidimensional space-time is believed to be relevant in the context of cosmology. Thorough study of Kaluza-Klein theory has been undertaken by Wesson [10]. A number of authors (refer [11–18] and references therein) have studied the physics of the universe in higher dimensional space-time.

The gravitational constant $G$ couples geometry to matter in the Einstein field equation, and in expanding universe it is considered as a function of time. A possible time variable $G$ was suggested by Dirac [19] and this has been extensively discussed in literature [20]. Since its introduction, its significance has been studied from time to time by various workers [21–23]. In modern cosmological theories, the cosmological constant remains a focal point of interest. A wide range of observations now compellingly suggest that the universe possesses a non-zero cosmological constant [24]. In the context of quantum field theory, a cosmological term corresponds to the energy density of vacuum. The birth of the universe has been attributed to an excited vacuum fluctuation triggering off an inflationary expansion followed by the supercooling. The release of locked up vacuum energy results in subsequent reheating. The cosmological term, which is measure of the energy of empty space, provides a repulsive force opposing the gravitational pull between the galaxies. If the cosmological term exits, the energy it represents counts as mass because mass and energy are equivalent. If the cosmological term is large enough, its energy plus the matter in the universe could lead to inflation. Unlike standard inflation, a universe with a cosmological term would expand faster with time because of the push from the cosmological term [25]. Dolgov [26, 27] and Sahni and Starobinsky [28] point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a “constant”. However, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying $\Lambda$ can be found. This entails that energy has to be conserved by a decrease in the energy density of the vacuum component followed by a corresponding increase in the energy density of matter or radiation (see also Weinberg [29] and Carroll et al. [30]).

Recent observations by Perlmutter et al. [31] and Riess et al. [32] strongly favour a significant and a positive value of $\Lambda$ with magnitude $\Lambda(G\hbar c^3) \approx 10^{-123}$. 
Their study is based on more than 50 type Ia supernovae with red-shifts in the range $0.10 \leq z \leq 0.83$ and these suggest Friedmann models with negative pressure matter such as a cosmological constant ($\Lambda$), domain walls or cosmic strings (Vilenkin [33], Garnavich et al. [34]). Recently, Carmeli and Kuzmenko [35] have shown that the cosmological relativistic theory predicts the value for cosmological constant $\Lambda = 1.934 \times 10^{-35} \text{ s}^{-2}$. This value of “$\Lambda$” is in excellent agreement with the recent estimates of the High-Z Supernova Team and Supernova Cosmological Project (Perlmutter et al. [31]; Riess et al. [32]; Garnavich et al. [34]; Schmidt et al. [36]). These observations suggest on accelerating expansion of the universe.

Bertolami [37] obtained cosmological models with time dependent $G$ and $\Lambda$ and suggested $\Lambda \sim R^{-2} - \tau^{-2}$. A number of authors [38–45] have proposed linking of the variation of $G$ with that of $\Lambda$ within the framework of general relativity. This approach is appealing because it leaves Einstein’s equations formally unchanged since a variation in $\Lambda$ is accompanied by the variation in $G$. Motivated by above studies, we thought it is worthwhile to consider cosmological models in higher dimensional space-time. These models might be more relevant since varying $G$, and $\Lambda$ play important role during early stages of the universe.

2. THE FIELD EQUATIONS

We shall begin by considering an isotropic and homogeneous higher dimensional universe represented by Robertson-Walkar like space-time metric

$$ds^2 = dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\chi_n^2 + \sin^2 \theta_1 d\theta_1^2 + \ldots + \sin^2 \theta_1 \sin^2 \theta_2 \ldots \ldots \sin^2 \theta_{n-1} d\theta_{n-1}^2) \right].$$

where

$$d\chi_n^2 = d\theta_1^2 + \ldots \ldots \sin^2 \theta_1 \sin^2 \theta_2 \ldots \ldots \sin^2 \theta_{n-1} d\theta_{n-1}^2,$$

Here $R(t)$, $k = 0, \pm 1$ and $D = n + 2$ stand for scale factor, curvature parameter and total number of space-time dimensions respectively.

The Einstein field equations with time varying cosmological and gravitational ‘constants’

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G[(\rho + p)u_\mu u_\nu - pg_{\mu\nu}] + \Lambda g_{\mu\nu},$$

for the metric (1) yield two independent equations,

$$\frac{n(n+1)}{2} \left[ \left( \frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} \right] = 8\pi G \rho + \Lambda,$$
Here equation (3) is time-time component and equation (4) the space-space component of the field equation (2). The over-dot denotes a derivative with respect to time co-ordinate \( t \). Further equations (3) and (4), provide the continuity equation

\[
\dot{\rho} + (n+1)(\rho + p) \frac{\dot{R}}{R} + \rho \frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0. \tag{5}
\]

From equation (5), it can be seen that in the present case the energy density of the matter fields is not conserved because of the varying character of coupling scalars \( G \) and \( \Lambda \). The principle of equivalence requires only \( g_{\mu\nu} \) (not \( \Lambda \) and \( G \)) should be involved in the equations of motion of particles and photons. Thus in the present case also the conservation law of energy-momentum \( (u_\mu T^{\mu\nu}; \nu = 0) \) holds and it suggests

\[
\dot{\rho} + (n+1)(\rho + p) \frac{\dot{R}}{R} = 0. \tag{6}
\]

With help of equation (6), equation (5) yields

\[
\dot{G} = -\frac{\dot{\Lambda}}{8\pi \rho}. \tag{7}
\]

Equation (7) plays key role in investigations of cosmological models.

Using equation of state \( p = (\gamma - 1)\rho \), equation (6) on integration produces

\[
\rho = C_1 R^{-(n+1)\gamma}, \tag{8}
\]

where \( C_1 = \rho_0 R_0^{-(n+1)\gamma} \) and suffix 0 represents the present value of the parameters.

Eliminating \( \dot{R}/R \) from equations (3) and (6), we obtain

\[
\frac{\rho^2}{\rho^3} = (n+1)^2 \gamma^2 \left[ \frac{16\pi G}{n(n+1)} + \frac{2\Lambda}{n(n+1)\rho} - \frac{k}{R^2 \rho^2} \right]. \tag{9}
\]

Again differentiating equation (9) and using equation (7), we obtain

\[
2 \frac{\dot{\rho}}{\rho} = 3 \left( \rho \frac{\dot{G}}{G} \right)^2 = (n+1)\gamma^2 \left[ \frac{(n+1)\gamma - 2k}{\gamma R^2} - \frac{2\Lambda}{n} \right]. \tag{10}
\]

If equation (8) be a solution of equation (10), \( R(t) \) must satisfy the differential equation
\[ 2\dot{H} + (n+1)\gamma H^2 + [(n+1)\gamma - 2]\frac{k}{R^2} - \frac{2\Lambda\gamma}{n} = 0, \quad (11) \]

where \( H = \dot{R}/R \) is the Hubble parameter.

### 3. SOLUTIONS AND THEIR ANALYSIS

#### 3.1. CASE I

Several authors have suggested that \( \Lambda \sim R^{-2} \), by invoking different assumptions [46–48]. Chen and Wu [46] have suggested that in the relation \( \Lambda = \alpha R^{-2} \), the constant \( \alpha \) is related to the curvature parameter \( k \) and hence we may assume

\[ \Lambda = C_2 R^{-2}, \quad (12) \]

where

\[ C_2 = \left[ \frac{n[(n+1)\gamma - 2]k}{2\gamma} \right]. \]

In the present case, we should note that \( C_2 \) takes different values in different phases as \( \gamma \) lies between \( 1 \leq \gamma \leq 2 \) in different phases.

Using the specific choice of \( \Lambda \) given by equation (12), the last two terms in equation (11) gets canceled and we have

\[ 2\dot{H} + (n+1)\gamma H^2 = 0, \quad (13) \]

which on integration gives

\[ R = \left[ \frac{(n+1)\gamma}{2} C_3 t \right]^{2/(n+1)\gamma}, \quad (14) \]

where \( C_3 \) is an integrating constant. By use of equation (14), equation (8) and (12) suggest expressions for energy density and cosmological term as

\[ \rho = \frac{4C_1}{[(n+1)\gamma C_3]^2 t^2}, \quad (15) \]

\[ \Lambda = C_2 \left[ \frac{(n+1)\gamma}{2} C_3 t \right]^{-4/(n+1)\gamma}. \quad (16) \]

Substituting the values of \( \rho \) and \( \Lambda \) from equations (15) and (16) in equation (7) and then integrating, we obtain
\[ G = C_4 + \frac{nk}{8\pi C_1} \left[ \frac{(n+1)\gamma}{2} C_3 t \right]^{2-\frac{4}{2(n+1)\gamma}}, \]  \hspace{1cm} (17)

where \( C_4 \) is the constant of integration. In order to satisfy all the field equations by new values of \( R, \rho, \Lambda, \) and \( G, \) we obtain a relation between constants as

\[ C_4 = \frac{n(n+1)C_3^2}{16\pi C_1}. \]  \hspace{1cm} (18)

From equation (14), one can calculate the age of the universe as

\[ t = \frac{2}{(n+1)\gamma H}. \]  \hspace{1cm} (19)

The deceleration parameter \( q \) for the present model takes the form

\[ q = -\frac{R\dot{R}}{R^2} = \frac{1}{2} [(n+1)\gamma - 2]. \]  \hspace{1cm} (20)

Equation (20) shows that deceleration parameter is constant in this model and depend on the dimensionality of the space-time.

The relation between temperature and energy density as \( T \sim \rho^{\gamma-1/\gamma} \) has been widely accepted in the literature. With help of equation (15), we get

\[ T \sim t^{-2+2/\gamma}. \]  \hspace{1cm} (21)

Considering age of the universe about \( t_0 \sim 10^{10} \) yrs \( \sim 3 \times 10^{17} \) seconds and \( \gamma = 4/3, \) equation (21) presents approximate value of temperature \( T_0 \sim 1 \) K, which is in fair agreement with measured value of thermal radiation in universe [49].

The horizon distance, \( d_H(t) \), at time \( t \) is the proper distance traveled by light emitted at \( t = t_e \)

\[ d_H(t) = R(t) \lim_{t \to 0} \left[ \int_{t_e}^t \frac{dt'}{R(t')} \right]. \]  \hspace{1cm} (22)

By use of equation (14), one can obtain the horizon distance

\[ d_H(t) = \frac{(n+1)\gamma}{(n+1)\gamma - 2} t, \]  \hspace{1cm} (23)

which shows that causal communication between two observers exist. Further equation (23) indicates that the horizon distance was smaller during early stages of the universe where \( n \) might be more than 2.
3.2. CASE II

In the spirit of quantum cosmology, Chen and Wu et al. [46] argued that as \( \Lambda \) has dimension of inverse length squared, one can express

\[
\Lambda \propto \frac{1}{l_{Pl}^2} \left[ \frac{l_{Pl}}{R} \right]^n, \quad (h = c = 1),
\]

where \( l_{Pl} \) is the Planck length. Here, the term \( h \) does not appear because general relativity is a classical theory. It is obvious to put \( n = 2 \) in the above expression. The relation \( \Lambda \propto R^{-2} \) does not fall in conflict with the high degree of isotropy of the cosmic background radiation. This ansatz was generalized by Carvalho et al. [50] by including a term proportional to the Hubble parameter. They have presented a decaying \( \Lambda \) of the form

\[
\Lambda = \alpha R^{-2} + \beta H^2. \tag{24}
\]

Here \( \alpha, \beta \) are adjustable dimensionless parameters. The additional term \( \beta H^2 \) can modify some features of cosmological models related to the age and low energy density problem. Making use of equation (24) in (11), we obtain

\[
2HH' + \left[ \frac{n(n + 1)\gamma - 2\beta \gamma}{n} \right] \frac{H^2}{R} + \left[ \frac{[(n + 1)\gamma - 2]}{\gamma} \right] \frac{2\alpha}{n} \frac{\gamma}{R^3} = 0, \tag{25}
\]

where prime denotes derivative with respect to scale factor \( R \). Using the transformation \( H^2 = U \), equation (25) reduces to Leibnitz linear differential equation

\[
U' + \gamma \left[ (n + 1) - \frac{2\beta}{n} \right] \frac{1}{R} U = \left[ \frac{2\alpha}{n} - \frac{[(n + 1)\gamma - 2]}{\gamma} \right] \frac{\gamma}{R^3}, \tag{26}
\]

which on integration yields

\[
H^2 = AR^{-2} \tag{27}
\]

with

\[
A = \frac{[2\gamma \alpha - (n + 1)\gamma - 2]nk}{[n(n + 1)\gamma - 2\beta \gamma - 2n]}.
\]

Further, on integrating equation (27) we obtain

\[
R = A^{1/2}t + C. \tag{28}
\]

Taking initial conditions \( i.e. \) the value of scale factor \( R = 0 \), when \( t = 0 \) at the time of origin of the universe, equation (28) suggest that constant of integration \( C \) must vanish and hence

\[
R(t) = A^{1/2}t. \tag{29}
\]
With help of equation (29), equations (8) and (24) take the form

\[ \rho = C_1 A^{-(n+1)\gamma/2} t^{-(n+1)\gamma'}, \]  

(30)

\[ \Lambda = (\alpha + A\beta) R^{-2} = \frac{\alpha + A\beta}{At^2}. \]  

(31)

Using equations (30) and (31) and integrating equation (7), we obtain

\[ G = G_0 + \left( \frac{(\alpha + A\beta) A^{[(n+1)\gamma-2]}/2}{4\pi C_1 ((n+1)\gamma - 2)} \right) t^{(n+1)\gamma-2}. \]  

(32)

From equation (31), it is clear that \( \Lambda \sim t^{-2} \), which is well accepted result in cosmology. The cosmological implications of a time varying cosmological constant of the form \( \Lambda \sim t^{-2} \) have been discussed by several authors [38, 41, 42]. In this case the deceleration parameter takes value zero as the scale factor goes linearly with time. This result also support the view of present day observations which suggest \( q = 0.3 \pm 0.3 \) [51]. Using equation (29), equation (22) gives the horizon distance

\[ d_H(t) = t \log(t). \]  

(33)

The temperature-density relation for present case suggest

\[ T \sim t^{-(n+1)(\gamma-1)}. \]  

(34)

Equation (34) suggest that the universe was hotter during early stages of the evolution of the universe.

3.3. CASE III

In most of the investigations a power law relation between the scale factor and scalar fields assumed. Cosmological models with the gravitational and cosmological constants generalized as coupling scalars where \( G \sim R^\alpha \) have been discussed by Sistero [40]. In this section we shall consider

\[ G = C_5 R^\alpha, \]  

(35)

where \( C_5 \) is the proportionality constant and \( \alpha \geq 0 \). Using the values of \( \rho \) and \( G \) given by equations (8) and (35), equation (7) reduces to

\[ \Lambda' = -8\pi\alpha C_1 C_5 R^{\alpha-(n+1)(\gamma-1)}. \]  

(36)

Here prime denotes derivative with respect to \( R \). Equation (36) produces the solution
where constant of integration is considered to be zero with the spirit that at beginning of the universe \( R = 0, \Lambda = 0 \).

From equations (3), (8), (35) and (37), we have

\[
\dot{R}^2 = \frac{16\pi C_1 C_2 \gamma}{n(n+1)\gamma - \alpha} R^{\alpha+2-(n+1)\gamma} - k. \tag{38}
\]

It is difficult to integrate equation (38) for a general \( k \). Under the assumption \( k = 0 \) for flat model equation (38) yields the solution

\[
R = B t^{2/[(n+1)\gamma - \alpha]}, \tag{39}
\]

where

\[
B = \left[ \frac{A((n+1)\gamma - \alpha)}{2} \right]^{2/[(n+1)\gamma - \alpha]}
\]

and

\[
A = \left[ \frac{16\pi C_1 C_2 \gamma}{n(n+1)\gamma - \alpha} \right]^{1/2}.
\]

It can be seen from equation (39) that for expanding model of the universe \((n+1)\gamma - \alpha < 2\). With help of the result (39), energy density \( \rho \), \( G \) and \( \Lambda \) may be obtained from equations (8), (35), and (37) respectively

\[
\rho = C_1 B^{-(n+1)\gamma} t^{-2[(n+1)\gamma - (n+1)\gamma - \alpha]}, \tag{40}
\]

\[
G = C_2 B^{\alpha[(n+1)\gamma - \alpha]}, \tag{41}
\]

\[
\Lambda = \frac{8\pi\alpha C_1 C_5}{(n+1)\gamma - \alpha} B^{(\alpha-(n+1)\gamma)} t^{-2}. \tag{42}
\]

From above equations, it can be easily seen that energy density and cosmological term \( \Lambda \) are decreasing while gravitational constant ‘G’ is increasing during the expansion of the universe. When the universe is required to have expanded from a finite minimum volume, the critical density assumption and conservation of energy-momentum tensor dictate that \( G \) increases in a perpetually expanding universe [38, 47]. In most variable \( G \) cosmologies [52, 53], \( G \) is a decreasing function of time. But the possibility of an increasing \( G \) has also been suggested by several authors [54].

In this case, the deceleration parameter \( q \), temperature \( T \) and horizon distance \( d_H(t) \) are given by

\[
q = -1 + \frac{1}{2} [(n+1)\gamma - \alpha], \tag{43}
\]
\[ T = t^{-2(n+1)(\gamma-1)/(n+1)\gamma-\alpha}, \]  
\[ d_H(t) = \left[ \frac{(n+1)\gamma-\alpha}{(n+1)\gamma-\alpha-2} \right] t. \]

It is interesting to note that for \((n+1)\gamma-\alpha < 2\) the deceleration parameter \(q < 0\), which is in good agreement with observations (Knop et al. [55], Riess et al. [56]). Recently redshift magnitude test has had a chequered history. Before the first indication of an accelerating universe on the basis of observational consequences by Perlmutter et al. [31] and Riess et al. [32], in 1998, the deceleration parameter \(q_0\) was claimed to lie between 0 and 1 leading to the suggestion that the universe is decelerating. But today, the view is entirely different. Observations of Type Ia Supernovae (SNe) [55, 56] allow us to probe the expansion history of the universe leading to the conclusion that the expansion of the universe is accelerating. Vishwakarma and Narlikar [57] and Virey et al. [58] have reviewed the determination of the deceleration parameter from Supernovae data. From (44) and (45) we can see that temperature of the universe is decreasing and horizon distance is finite.

4. CONCLUSIONS

In this work we have presented a new class of cosmological models with varying gravitational coupling \(G\) and cosmological term \(\Lambda\) in higher dimensions which yield several four-dimensional results when \(n = 2\). The analytic expressions of cosmological variables indicate a clear dependence of dynamical properties on dimensionality of the space-time. If the horizon distance defined by (22) is finite throughout, then we can say that the causal communication between two observers exist. It can be easily seen that horizon distance is finite in all models. The boundary of horizon is smaller during higher dimensions. The results obtained in this paper are in favour of the views of astronomical observations. The deceleration parameter and temperature are well within the observational limits.

The behaviour of the universe in models will be determined by the cosmological term \(\Lambda\); this term has the same effect as a uniform mass density \(\rho_{\text{eff}} = -\Lambda/4\pi G\), which is constant in space and time. A positive value of \(\Lambda\) corresponds to a negative effective mass density (repulsion). Hence, we expect that in the universe with a positive value of \(\Lambda\), the expansion will tend to accelerate; whereas in the universe with negative value of \(\Lambda\), the expansion will slow down, stop and reverse. The observations on magnitude and red-shift of type Ia supernova suggest that our universe may be an accelerating one with induced cosmological density through the cosmological \(\Lambda\)-term. Thus our models are consistent with the results of recent observations.
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