BEHAVIOR OF THREE QUANTUM RADIATORS IN MICROCAVITY*

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The behavior of the system of radiators at short and long time intervals in comparison with the retardation between them is studied. The entanglement behavior of atomic states in the process of spontaneous emission is determined. The theory that describes the spontaneous emission of two hydrogen-like atoms within a short time period in comparison with the inverse value of photon emission frequency is proposed. It is demonstrated that at a short time interval the rate of spontaneous emission in an oscillatory manner tends to the exponential law of spontaneous emission. The simple kinetic equation, which describes this stage of system evolution, is obtained.

Key words: quantum optics, quantum information, spontaneous emission, retardation effects and causality.

1. INTRODUCTION

The interaction of the collective of two level radiators with cavity electromagnetic field was in the center of attention in the many experimental and theoretical studies published recently [1–3]. This is connected with the large application of two-level system as a qubit in quantum computing and quantum processing of information. In many cases it is used the distinguished ensembles of qubits in the realization of quantum states of registers. On the other hand, the Bose–Einstein condensations of atomic ensembles give us the possibility to regard two-level atoms like an undistinguished ensemble [2, 3]. In this representation, the N two-level atoms have a more reduced number of collective states. For example, four atomic states for two atoms, A and B, are reduced to three states [4, 5]. This effect takes place because two excitations, atom A in excited state and B in ground state and atom A in ground state and B in excited state respectively, is regarded as a single collective state. In a general case, N two-level atoms with the number of states $2^N$ can be reduced to $N + 1$ states in


processes of coherent excitation in accordance with the principle of indistinguishability between the radiators.

In the modern experiment the new quantum phenomena in the radiation interaction of single atoms with single mode maser or micro masers was studied [6]. Trapping states in the one-atom maser or micromaser are predicted by the maser theory and are a direct consequence of the quantization of the electromagnetic (EM) field. This effect has been observed in the experimental work of the late professor H. Wather [7]. Recently, by integrating the techniques of laser cooling and trapping with those of cavity quantum electrodynamics, an ensemble of atoms trapped within the mode of an optical cavity was determined in real time by monitoring the transmission of a weak probe beam [8].

In this paper the behavior of ensemble consisted from three undistinguished atoms in interaction with one mode of microcavity is discussed. This problem is reduced to the solution of characteristic equation for the linear system, which contains $N + 1$ equations. The method developed for $N + 1$ equations, don’t give us the possibility to represent analytically the solution of system of atoms in interaction with cavity EM field. We reduced the number of $2^3$ states, for three two-level systems in interaction with the cavity field, to four levels and solved exactly the system of linear equations for four operators of EM field.

2. FOUR LEVEL MODEL FOR DESCRIPTION OF THREE UNDISTINGUISHED ATOMS

Let us consider the system consisted from three undistinguished atoms obtained by using cooling technique [8], prepared in coherent state. The scheme of further interaction with cavity field is represented in Fig. 1. Flying through the Ramsay zone the system of atoms is prepared in the coherent state by the classical field. Considering that the coherent state of system of atoms is formed in ground state $|g\rangle = |g_1\rangle \otimes |g_2\rangle \otimes |g_3\rangle$, we can use the Dicke representation [7] of these radiators with total angular momentum $j = 1/2 + 3 = 3/2$. In this case, the undistinguished three-spin-ensemble is described by four states represented by collective excitation and annihilation operators $J^+ = \sqrt{3}|4\rangle\langle 3| + 2|3\rangle\langle 2| + \sqrt{3}|2\rangle\langle 1|$, $J^- = \sqrt{3}|3\rangle\langle 4| + 2|2\rangle\langle 3| + \sqrt{3}|1\rangle\langle 2|$, satisfy the commutation relations $[J^+, J^-] = 2J_z$, $[J_z, J^\pm] = \pm 2J^\pm$.

Here $J_z = \frac{1}{2}(3|4\rangle\langle 4| + 3|3\rangle\langle 3| - 2|2\rangle\langle 2| - 3|1\rangle\langle 1|)$ is the collective inversion operator. The collective states $|1\rangle_a, |2\rangle_a, |3\rangle_a$ and $|4\rangle_a$ are represented through the angular momentum states $|j, m\rangle$, $-j \leq m \leq j$, $|3/2, -3/2\rangle = |1\rangle_a$, $|3/2, 1/2\rangle = |2\rangle_a$. 

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are ground, first intermediate, second intermediate and excited states respectively (Fig. 1).

Fig. 1 – The collective states of three undistinguished atoms. In the time moment \( t = 0 \) the atoms entered the cavity and during the time interval \( t_0 \) the atoms intensively interact with single mode of cavity field represented in the scheme.

According to this representation, we observe that 2\(^3\) states are reduced to four collective atomic states. In other words, if we prepare the three undistinguished atoms in coherent state with external laser field (see for example [9])

\[
|\mu\rangle = \exp(\mu J^+) - |j,j\rangle \left(1 + |\mu|^2\right)^{-j}, \quad j = 3/2
\]

one can reduce the initial atomic state to a superposition of four states

\[
|\mu\rangle = \left[|1\rangle_a + \mu \sqrt{3} |2\rangle_a + \mu^2 \sqrt{3} |3\rangle_a + \mu^3 |4\rangle_a \right] \left(1 + |\mu|^2\right)^{-3/2}.
\]

Let us consider the initial time, \( t \), the time for which the atomic ensemble reached the cavity. After this time the interaction with resonator electromagnetic field begins. In the initial time moment, \( t \), the atomic and cavity field states are totally factorized, so that this state can be represented

\[
|\Psi(t)\rangle = |\Psi_A\rangle \otimes |\Psi_{CEM}\rangle = \\
\exp(-iH_0 t) \left[\alpha |4\rangle_a + \beta |3\rangle_a + \gamma |2\rangle_a + \delta |1\rangle_a \right] \otimes \sum_{n=\mathbb{N}_a} \mathcal{S}_n |n\rangle
\]

(1)

where \( H_0 = \hbar \omega J_z + \hbar \omega a^+ a \) is the free part of Hamiltonian for atomic and field subsystems, \( \alpha = \mu^3 \left(1 + |\mu|^2\right)^{-3/2} \), \( \beta = \mu^2 \sqrt{3} \left(1 + |\mu|^2\right)^{-3/2} \), \( \gamma = \mu \sqrt{3} \left(1 + |\mu|^2\right)^{-3/2} \),
\[ \delta = \left(1 + |\mu|^2\right)^{3/2}, \quad S_n \left(N_a \geq n \geq N\right) \] are the decomposition coefficients for cavity electromagnetic field on the Fock states \(|n\rangle\).

Considering that in the time moment \(t + \tau\) the interaction between atomic subsystem and cavity field begins, we describe this process with Schrödinger equation in interaction picture

\[ i\hbar \frac{\partial}{\partial \tau} \left| \Psi(t + \tau) \right\rangle = H_i \left| \Psi\left((t + \tau)\right)\right\rangle, \tag{2} \]

where \(H_i = \hbar \lambda [J^+a^+ + J^-a]\) is the interaction Hamiltonian which commutes with the free part of the Hamiltonian \([H_i, H_0] = 0\). In the time moment, \(t + \tau\), the state vector of coupled system, atoms and cavity field, can be represented through four electromagnetic field operators

\[ \left| \Psi\left((t + \tau)\right)\right\rangle = \exp(-iH_i \tau / \hbar) \left| \Psi(t)\right\rangle = (\alpha \tilde{X}(\tau) + \beta \tilde{Y}(\tau) + \gamma \tilde{Z}(\tau) + \delta \tilde{U}(\tau)) \sum_{n=N_a}^{N_0} S_n |n\rangle, \]

which can be represented using the evolution operator \(U(\tau) = \exp(-iH_\gamma \tau / \hbar)\)

\[ \dot{\tilde{X}}(\tau) = \exp\left(-iH_\gamma \frac{\tau}{\hbar}\right) |4\rangle_a, \quad \dot{\tilde{Y}}(\tau) = \exp\left(-iH_\gamma \frac{\tau}{\hbar}\right) |3\rangle_a, \]

\[ \dot{\tilde{Z}}(\tau) = \exp\left(-iH_\gamma \frac{\tau}{\hbar}\right) |2\rangle_a, \quad \dot{\tilde{U}}(\tau) = \exp\left(-iH_\gamma \frac{\tau}{\hbar}\right) |1\rangle_a. \tag{3} \]

In according with Schrödinger equation (2) and properties of collective atomic operators, \(J^+ |j,m\rangle = \sqrt{(j+1)(j-m)} |j,m+1\rangle, J^- |j,m\rangle = \sqrt{(j-m+1)(j+m)} |j,m-1\rangle, J^+ |4\rangle = 0, J^- |1\rangle = 0\), one can obtain the following system of equations for operator-functions \(X(\tau), Y(\tau), Z(\tau)\) and \(U(\tau)\).

\[ \frac{d\tilde{X}(\tau)}{d\tau} = i\lambda \sqrt{3}\tilde{Y}(\tau)a^+, \]

\[ \frac{d\tilde{Y}(\tau)}{d\tau} = i\lambda \{\sqrt{3}\tilde{X}(\tau)a + 2\tilde{Z}(\tau)a^+\}, \]

\[ \frac{d\tilde{Z}(\tau)}{d\tau} = i\lambda \{2\tilde{Y}(\tau)a + \sqrt{3}\tilde{U}(\tau)a^+\}, \]

\[ \frac{d\tilde{U}(\tau)}{d\tau} = i\lambda \sqrt{3}\tilde{Z}(\tau)a. \tag{4} \]

This system of equations cannot be solved using traditional Wronskian method proposed in the literature, because the functions and their coefficients are operators. As the operators in numerical representation are describes by matrix
elements, the computer simulation of such system becomes difficult with increasing the number of atoms and Fock states of the field. The analytical solution of this system can be found taking into account the ordering procedure in operator functions and its coefficients. After four derivations of equation for operator-function \( X(\tau) \) it is obtained the following differential equation

\[
\frac{d^4 \hat{X}(\tau)}{d\tau^4} + 10\lambda^2 \frac{d^2 \hat{X}(\tau)}{d\tau^2} (\hat{n} + 2) + 9\lambda^4 \hat{X}(\tau)(\hat{n} + 1)(\hat{n} + 3) = 0.
\]

Here \( \hat{n} = a^+ a \) is the photon number operator. The characteristic equation for this system of equations

\[
\Theta^4 + 10\lambda^2 (n + 2) \Theta^2 + 9\lambda^4 (n + 1)(n + 3) = 0,
\]

\[
\Theta_{1,2} (n + 2) = \pm i\Omega_1 (n + 2), \quad \Theta_{3,4} (n + 2) = \pm i\Omega_2 (n + 2)
\]

can be solved exactly.

The time behavior of the system is described by two quantum Rabi frequencies \( \Omega_1 (n + 2) \) and \( \Omega_2 (n + 2) \)

\[
\Omega_1 (n + 2) = \lambda \sqrt{5(n + 2) + \sqrt{16(n + 2)^2 + 9}},
\]

\[
\Omega_2 (n + 2) = \lambda \sqrt{5(n + 2) - \sqrt{16(n + 2)^2 + 9}}.
\]

(5)

Taking into account the initial conditions for operator functions \( X(0) = |4\rangle, Y(0) = |3\rangle, Z(0) = |2\rangle, U(0) = |1\rangle \), the solution of Schrödinger equation can be represented in the following form

\[
|\Psi(t + \tau)\rangle = \sum_{n=N_i}^{N_f} S_n \begin{cases}
\alpha \Phi(n+2) \left[ \frac{3\lambda^2(n+1) - \Omega_2^2[n+2]}{[n+2][\Omega_1^2(n+2)]} \right] \cos \left[ \frac{\Omega_1(n+2)\tau}{n} \right] |n\rangle + \\
+\beta \frac{1}{n^{3/2}} \Phi(n+1) \left\{ \frac{(\Omega_2^2[n+1] - 3\lambda^2 n)\Omega_1[n+1] \sin \Omega_1[(n+1)\tau]}{\Omega_2^2[n+1] - 3\lambda^2 n} \right\} \sqrt{n} |n-1\rangle + \\
+\gamma 2\sqrt{3}\lambda^2 \Phi(n) \left[ \cos \left[ \frac{\Omega_1(n)\tau}{n} \right] - \cos \left[ \frac{\Omega_2(n)\tau}{n} \right] \right] \sqrt{n(n-1)(n-2)} |n-2\rangle + \\
+i\delta 6\lambda^3 \Phi(n-1) \left[ \frac{1}{\Omega_1[n-1]} \sin \left[ \frac{\Omega_1(n-1)\tau}{n} \right] \right] \sqrt{n(n-1)(n-2)(n-3)} |n-3\rangle + \\
\end{cases}
\]
\[ 
\begin{align*}
&+\sum_{n=N_1}^{N_2} S_n \left[ \frac{1}{\Omega_1[n+2]} \sin[\Omega_1(n+2)\tau] \right] \\
&+\sum_{n=N_1}^{N_2} S_n \left[ \frac{1}{\Omega_2[n+2]} \sin[\Omega_2(n+2)\tau] \right] \\
&+\gamma \frac{1}{\sqrt{3}} \Phi(n) \left[ \left( \Omega_1[n] - \Omega_2[n] \right) \Omega_1[n+2] \sin[\Omega_1(n+2)\tau] \right] \\
&+\delta \Phi(n) \left[ \left( \Omega_1[n+2] - \Omega_2[n+2] \right) \Omega_1[n+2] \sin[\Omega_1(n+2)\tau] \right] \\
&+\beta \Phi(n) \left[ \frac{1}{\Omega_1[n+2]} \sin[\Omega_1(n+2)\tau] \right] \\
&+\beta \Phi(n) \left[ \frac{1}{\Omega_2[n+2]} \sin[\Omega_2(n+2)\tau] \right] \\
&+\beta \Phi(n) \left[ \frac{1}{\Omega_1[n+2]} \sin[\Omega_1(n+2)\tau] \right] \\
&+\beta \Phi(n) \left[ \frac{1}{\Omega_2[n+2]} \sin[\Omega_2(n+2)\tau] \right] \\
&+\beta \Phi(n) \left[ \frac{1}{\Omega_1[n+2]} \sin[\Omega_1(n+2)\tau] \right] \\
&+\beta \Phi(n) \left[ \frac{1}{\Omega_2[n+2]} \sin[\Omega_2(n+2)\tau] \right]
\end{align*} \]
The solution of Schrödinger equation satisfies the initial conditions necessary for the system of equations (4). The time behavior of atomic inversion and photon statistics of cavity electromagnetic field is discussed in the next section.

3. NUMERICAL BEHAVIOR OF ATOMIC INVERSION AND PHOTON NUMBER FLUCTUATIONS

In the case when the atomic ensemble interacts with the vacuum of the EM field the reversible energy exchange between the excited radiators and the vacuum of the EM field is possible. Let us consider the particular situation, when $\alpha = S_0 = 1$ and $\beta = \gamma = \delta = S_1 = S_2 = \ldots = S_n = \ldots = 0$. In this case the inverted ensemble of atoms interacts only with the cavity vacuum of the EM field. The solution of Schrödinger equation is

$$\Psi(t + \tau) = \left\{ \begin{array}{ll}
\frac{1}{2\sqrt{\gamma}} & \left( \sqrt{\gamma} - 7 \right) \cos \left[ \sqrt{10 + \sqrt{\gamma}} \tau \right] + \\
+ & (7 + \sqrt{\gamma}) \cos \left[ \sqrt{10 - \sqrt{\gamma}} \tau \right] \left\{ 0 \right\}_a + \\
+ & \frac{i\sqrt{\gamma}}{2\sqrt{\gamma}} \left( \sqrt{\gamma} + 1 \right) \sin \left[ \sqrt{10 + \sqrt{\gamma}} \tau \right] + \\
+ & (\sqrt{\gamma} - 1) \frac{1}{\sqrt{10 - \sqrt{\gamma}}} \sin \left[ \sqrt{10 - \sqrt{\gamma}} \tau \right] \left\{ -1 \right\}_a + \\
+ & \sqrt{\gamma} \frac{1}{\sqrt{10 + \sqrt{\gamma}}} \cos \left[ \sqrt{10 + \sqrt{\gamma}} \tau \right] - \\
+ & \frac{1}{10 - \sqrt{\gamma}} \sin \left[ \sqrt{10 - \sqrt{\gamma}} \tau \right] \sqrt{6} |3\rangle |1\rangle_a.
\end{array} \right.$$  

(7)

The number of excitations in the i-state is $\hat{N}_i = |i\rangle \langle i|$ and using the wave function (7) one can find the mean value of this operator $\langle \hat{N}_i \rangle = \langle \Psi(t + \tau) | i \rangle \langle i | \Psi(t + \tau) \rangle$. According to this definition we obtain the following expression for atomic population numbers
\[ \langle N_4 \rangle = \left( \frac{1}{2\sqrt{73}} \left[ (\sqrt{73} - 7) \cos \left( \sqrt{10 + \sqrt{73}} \lambda \tau \right) + \left( 7 + \sqrt{73} \right) \cos \left( \sqrt{10 - \sqrt{73}} \lambda \tau \right) \right] \right)^2, \]

\[ \langle N_3 \rangle = \left( \frac{\sqrt{3}}{2\sqrt{73}} \left[ (\sqrt{73} + 1) \frac{1}{\sqrt{10 + \sqrt{73}}} \sin \left( \sqrt{10 + \sqrt{73}} \lambda \tau \right) + \frac{\sqrt{3}}{\sqrt{10 - \sqrt{73}}} \sin \left( \sqrt{10 - \sqrt{73}} \lambda \tau \right) \right] \right)^2, \]

\[ \langle N_2 \rangle = \left( \frac{\sqrt{6}}{\sqrt{73}} \left[ \cos \left( \sqrt{10 + \sqrt{73}} \lambda \tau \right) - \cos \left( \sqrt{10 - \sqrt{73}} \lambda \tau \right) \right] \right)^2, \]

\[ \langle N_1 \rangle = \left( \frac{3\sqrt{2}}{\sqrt{73}} \left[ \frac{1}{\sqrt{10 + \sqrt{73}}} \sin \left( \sqrt{10 + \sqrt{73}} \lambda \tau \right) - \frac{1}{\sqrt{10 - \sqrt{73}}} \sin \left( \sqrt{10 - \sqrt{73}} \lambda \tau \right) \right] \right)^2. \]

The quantum properties of generating EM field in the cavity are described by normalized quadratic fluctuations \( \Delta^2 \) of photon number operator, \( n = a^*a \). Using the wave function (7) we obtained the following expression for quantum fluctuations \( \Delta^2 \)

\[ \Delta^2 = \langle a^*a^*aa \rangle - \langle a^*a \rangle^2 = \frac{324}{73} \left( \frac{1}{\sqrt{10 + \sqrt{73}}} \sin \left( \sqrt{10 + \sqrt{73}} \lambda \tau \right) - \frac{1}{\sqrt{10 - \sqrt{73}}} \sin \left( \sqrt{10 - \sqrt{73}} \lambda \tau \right) \right)^2 + \]

\[ + \frac{12}{73} \left( \cos \left( \sqrt{10 + \sqrt{73}} \lambda \tau \right) - \cos \left( \sqrt{10 - \sqrt{73}} \lambda \tau \right) \right)^2 - \]

\[ - \frac{162}{73} \left( \frac{1}{\sqrt{10 + \sqrt{73}}} \sin \left( \sqrt{10 + \sqrt{73}} \lambda \tau \right) - \frac{1}{\sqrt{10 - \sqrt{73}}} \sin \left( \sqrt{10 - \sqrt{73}} \lambda \tau \right) \right) + \]

\[ + \frac{12}{73} \left( \cos \left( \sqrt{10 + \sqrt{73}} \lambda \tau \right) - \cos \left( \sqrt{10 - \sqrt{73}} \lambda \tau \right) \right) + \]

\[ + \frac{3}{292} \left( \frac{\sqrt{73} + 1}{\sqrt{10 + \sqrt{73}}} \sin \left( \sqrt{10 + \sqrt{73}} \lambda \tau \right) + \frac{\sqrt{73} - 1}{\sqrt{10 - \sqrt{73}}} \sin \left( \sqrt{10 - \sqrt{73}} \lambda \tau \right) \right)^2. \]

The behavior of atomic populations on the collective states, as a function of trapping time \( \tau \), is represented in Fig. 2.
Fig. 2 – The dependence of mean number of atoms on the a) first level; b) second level; c) third level, and d) forth level as a function of flying time, for the following values of the parameters $\beta = \delta = \gamma = 0$, $\alpha = 1$. 
From the analysis of numerical results it follows that in the processes of collapses and revivals of the atomic inversion the atoms pass through the ground state $|1\rangle$. If we consider flying time equal to the time moment, $\tau$, for which the ensemble of atoms is in the ground state $|1\rangle$, one can observe that all photon energy, $3\hbar\omega$, remains in the cavity. These time moments correspond to collective maser condition, for which the inverted atomic system leaves the cavity in the ground state.

The behavior of quantum fluctuations of photon numbers is presented in Fig. 3. From the numerical simulation it follows the existence of the time intervals, for which the normalized square fluctuation takes negative values. This effect corresponds to the situation when the quantum noise of generated photons in the cavity is less than the noise in quantum electromagnetic vacuum.

**Fig. 3** – The normalized square fluctuations of photon number operator in the cavity.

### 4. CONCLUSIONS

From analytical and numerical results it follows the existence of the moment of time during which the ensemble of atoms comes back to its initial separated state (1). In a situation when these time moments coincide with the flying time through the cavity, the ensemble of atoms leaves the cavity in the excited state, and EM field from the cavity remains in the vacuum state. This case corresponds to the full disentanglement between the cavity field and trapped atomic ensemble.

It is possible to realize the micro maser regimes for three atoms, choosing the flying time for which the radiators pass totally in the ground state. The realization of such regimes is possible in the good cavity limits.
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