APPLICATIONS OF THE JACOBI GROUP TO QUANTUM MECHANICS

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Received August 30, 2008

Infinitesimal holomorphic realizations for the Schrödinger-Weil representation and the discrete series representations of the Jacobi group are constructed. Explicit expressions of the basic differential operators are obtained. The squeezed states for the unitary irreducible representation of the Jacobi group are introduced. Matrix elements of the squeezed operators, expectation values of polynomial operators in infinitesimal generators of the Jacobi group, the squeezing region and a description of Mandel’s parameter are presented.

1. INTRODUCTION

The Jacobi group $G_J$ is the semidirect product of the symplectic group $Sp(n, \mathbb{R})$ with an appropriate Heisenberg group [1–3]. In [4–6] we have considered the Perelomov coherent states and the squeezed states for the Jacobi group. The representation theory of the Jacobi group has been constructed in [2, 3, 7–9] with relevant topics: Schrödinger-Weil and metaplectic subrepresentations, classification and realizations of irreducible unitary representations over local fields, holomorphic Jacobi forms and automorphic representations, symplectic orbits, Whittaker models, Hecke and generalized Kac-Moody algebras, L-functions and modular forms, spherical functions, and the ring of invariant differential operators.

In Section 2.1 we review briefly some basic facts about the Jacobi group $G_J = G^J$. We realize the infinitesimal generators of $G_J$ in terms of boson operators and standard infinitesimal generators of the symplectic group. In Section 2.2 we construct infinitesimal holomorphic realizations for the Schrödinger-Weil representation and the discrete series representations. In Section 3 we introduce appropriate squeezed states for these representations.
2. UNITARY REPRESENTATIONS OF THE JACOBI GROUP

2.1. THE FUNDAMENTAL PRINCIPLE

The real Jacobi group $G^J$ is a subgroup of the symplectic group $Sp(2, \mathbb{R})$ consisting of $4 \times 4$ real matrices $g = (\lambda, \mu, \kappa, M)$ of the form

$$g = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad ad - bc = 1 ,$$

(1)

where $(\lambda, \mu, \kappa) \in H(\mathbb{R})$ and $M \in SL(2, \mathbb{R})$ [2]. Here $H(\mathbb{R})$ is the three-dimensional real Heisenberg group and $SL(2, \mathbb{R}) = Sp(1, \mathbb{R})$ is the special linear group. Then $G^J$ is the semidirect product of $H(\mathbb{R})$ with $SL(2, \mathbb{R})$. Let $\{X_1, ..., X_n\}_F$ denote the Lie algebra over $F$ with the basis elements $X_1, ..., X_n$. We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ the field of real numbers, the field of complex numbers, the ring of integers, and the set of non-negative integers, respectively. The Lie algebras of $G^J$, $H(\mathbb{R})$, and $SL(2, \mathbb{R})$ are denoted by $\mathfrak{g}^J = \{P, Q, R, F, G, H\}_F$, $\mathfrak{h} = \{P, Q, R, F, G, H\}_F$, $\mathfrak{sl}(2, \mathbb{R}) = \{F, G, H\}_F$, respectively. $P$, $Q$, $R$, $F$, $G$, $H$ are $4 \times 4$ matrices of coefficients $F_{ij} = \delta_{i1}\delta_{j3} + \delta_{i3}\delta_{j1}$, $G_{ij} = \delta_{i2}\delta_{j3} - \delta_{i3}\delta_{j2}$, $H_{ij} = \delta_{i1}\delta_{j1} - \delta_{i3}\delta_{j3}$, $P_{ij} = \delta_{i2}\delta_{j1} - \delta_{i3}\delta_{j4}$, $Q_{ij} = \delta_{i1}\delta_{j4} + \delta_{i2}\delta_{j3}$, $R_{ij} = \delta_{i2}\delta_{j4}$, where $i, j = 1, 2, 3, 4$. We get the commutators [2]


(2)


all other are zero. The center of $G^J$, consisting of all $(0, 0, \kappa) \in H(\mathbb{R})$, is thus isomorphic to $\mathbb{R}$. Every non-trivial central character $\psi$ of index $m \in \mathbb{R}$ can be obtained as $\psi((0, 0, \kappa)) = \exp(2\pi im\kappa)$, where $(0, 0, \kappa) \in H(\mathbb{R})$.

Let $\pi$ an unitary irreducible representation of $G^J$ of nonzero index $m$ on a complex separable Hilbert space $\mathcal{H}$. Let $\hat{\pi}$ be the derived representation of $\pi$ and let $\mathcal{D}$ be the space of smooth vectors. Denote $X = \hat{\pi}(X)$ for any $X \in \mathfrak{g}^J$. Then $\hat{\pi}(R) = i\mu I$, where $\mu = 2\pi m$ and $I$ is the identity operator. We now introduce the following operators in the complexification $\hat{\pi}(\mathfrak{g}^J_C)$ of $\hat{\pi}(\mathfrak{g}^J)$:
where $\sigma = \mu/|\mu|$. Using (2) and $X^\dagger = -X$ for $X \in \hat{\mathfrak{g}}^J$, we obtain

**Proposition 1.** $\hat{\mathfrak{g}}^J = \{ I, a, a^\dagger, K_+, K_-, K_0 \}_C$ with $(a^\dagger)^2 = a$, $K_\pm^2 = K_\pm$, $[a, K_+] = a^\dagger$, $[K_-, a] = 0$, $2[a, K_0] = a$, and

$$[a, a^\dagger] = I, \quad [K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0.$$ (4)

Then $\hat{\mathfrak{g}}(\mathfrak{sl}(2, \mathbb{R})) = \{ K_0, K_1, K_2 \}_\mathbb{R}$, where $K_\pm = K_1 \pm iK_2$. Consider the following operators in the universal enveloping algebra $\mathcal{U}(\hat{\mathfrak{g}}^J)$ [2]:

$$W_- = K_- - \frac{1}{2}a^2, \quad W_+ = K_+ - \frac{1}{2}(a^\dagger)^2, \quad W_0 = K_0 - \frac{1}{2}a^\dagger a - \frac{1}{4}I.$$ (5)

We have $[W_0, W_\pm] = \pm W_\pm$, $[W_-, W_+] = 2W_0$, $W_\dagger = W_\pm$ and $[a, W_\pm] = 0$, where $\sigma = 0, +, -$. Let $\mathfrak{w} = \{ W_0, W_1, W_2 \}_\mathbb{R}$, where $W_\pm = W_1 \pm iW_2$. The Casimir operator of $\mathfrak{w} = \mathfrak{sl}(2, \mathbb{R})$ is defined by $C = W_0^2 - W_1^2 - W_2^2$. The metaplectic group $Mp(2, \mathbb{R})$ is the non-split two-fold cover of $SL(2, \mathbb{R})$.

Using the Lie algebra $\mathfrak{w}$, the representation theory of $G^J$ may be fully reduced to Waldspurger’s representation theory of $Mp(2, \mathbb{R})$ [2, 10]. The Stone-von Neumann theorem and the method of Mackey for semidirect products lead us to the fundamental principle in the representation theory of the Jacobi group [2]:

Any representation $\pi$ of $G^J$ with index $m \neq 0$ is obtained in a unique way as $\pi = \pi^m_{SW} \otimes \hat{\pi}$, where the Schrödinger-Weil representation $\pi^m_{SW}$ is a certain projective representation of $G^J$ and $\hat{\pi}$ is a representation of the metaplectic group $Mp(2, \mathbb{R})$ (considered as a projective representation of $SL(2, \mathbb{R})$). The representations $\pi$ and $\hat{\pi}$ are simultaneously unitary, and irreducible.

The irreducible unitary representation $\pi$ of $G^J$ of index $m \neq 0$ are infinitesimally equivalent to the principal series representations $\hat{\pi}_{m\nu}$ for $s \in i\mathbb{R} \cup (-1/2, 1/2)$, $\nu = \pm 1/2$, with $C = (s^2 - 1)/4$, or to the positive and negative discrete series representations $\hat{\pi}^\pm_{mk}$ for $k \in \mathbb{Z}$, $k \geq 1$, with $C = (k - 1/2)(k - 5/2)/4$ [2]. Here $2K_0$ has the integral dominants weights $k$ for $\hat{\pi}^+_{mk}$ and $1 - k$ for $\hat{\pi}^-_{mk}$. 

\[
a = \frac{1}{2\sqrt{|\mu|}}(P - i\sigma Q), \quad a^\dagger = -\frac{1}{2\sqrt{|\mu|}}(P + i\sigma Q),
\]
\[
K_\pm = \mp \frac{1}{2}H - \frac{i\sigma}{2}(F + G), \quad K_0 = \frac{i\sigma}{2}(G - F),
\]

(3)
Standard models of the preceding representations are presented in [2]. Let \( L^2_{\text{hol}}(\mathcal{M}, \nu) \) denote the complex Hilbert space of all \( \mathbb{C} \)-valued holomorphic functions on the complex manifold \( \mathcal{M} \) which are square-integrable with respect to the measure \( \nu \). Consider the upper half-plane \( \mathbb{H} \) of all \( \tau \in \mathbb{C} \) with \( \Im \tau > 0 \). Suppose \( k > 3/2 \) and \( m \neq 0 \). Then there is an irreducible unitary representation \( \pi_{mk} \) on \( L^2_{\text{hol}}(\mathbb{C} \times \mathbb{H}, \mu_{mk}) \) with the Petersson measure \( d\mu_{mk} = \exp\left(-4\pi i y^2/\nu\right) y^{k-3} d^2 z d^2 \tau \), where \( y = \Re z \) and \( \nu = \Re \tau \) [2, 7, 9]. For any \( f \in L^2_{\text{hol}}(\mathbb{C} \times \mathbb{H}, \mu_{mk}) \) and \( g = \left((\lambda, \mu, \kappa), M\right) \in G^J \), we have

\[
\pi_{mk}(g^{-1})f(z, \tau) = (e^T + d)^{-k} \exp\left[2\pi i m(\kappa + \theta)\right]f(z_g, \tau_g),
\]

where \( (z, \tau) \in \mathbb{C} \times \mathbb{H}, \quad z_g = (e^T + d)^{-1}(z + \lambda^T + \mu), \quad \tau_g = (e^T + d)^{-1}(\alpha T + b), \) and \( \theta = \lambda z + (\lambda z_g - c z_g^2)(e^T + d) \). The Jacobi forms [1] are associated with \( \pi_{mk} \) provided the index \( m \) and the weight \( k \) are positive integers. \( \pi_{mk} \) can also be used to produce bases for signal processing, continuous windowed Fourier and wavelet transforms [11]. The group \( G^J \) is unimodular. The representation \( \pi_{mk} \) is square-integrable modulo the center of \( G^J \). Then \( \pi_{mk} \) is a Perelomov coherent state representation based on \( \mathbb{C} \times \mathbb{H} \) [11]. Let \( \mathbb{D} \) be the open disk of the points \( w \in \mathbb{C} \) with \( |w| < 1 \). The manifolds \( \mathbb{C} \times \mathbb{H} \) and \( \mathbb{C} \times \mathbb{D} \) can be biholomorphically identified by a partial Cayley transform [2]. Then there is an irreducible unitary representation \( \rho_{mk} \) on \( L^2_{\text{hol}}(\mathbb{C} \times \mathbb{D}, \nu_{mk}) \) which is unitarily equivalent to \( \pi_{mk} \) [2, 7].

### 2.2. INFINITESIMAL REPRESENTATIONS

#### 2.2.1. Holomorphic realizations

Although it is a non-reductive algebraic group, \( G^J \) can be considered as a group of Harish-Chandra type [2]. Then the complexification of \( \hat{\pi}(\mathfrak{g}^J) \) is the direct sum of vector spaces \( \hat{\pi}(\mathfrak{g}^J) = \mathfrak{p}_+ + \mathfrak{t} + \mathfrak{p}_- \) with \( \mathfrak{p}_- = \mathfrak{p}_+^\perp, \quad [\mathfrak{t}, \mathfrak{p}_-] \subset \mathfrak{p}_- \), where \( \mathfrak{p}_+ = \{a^+, K_+\}_\mathbb{C} \), and \( \mathfrak{t} = \{I, K_0\}_\mathbb{C} \). Consider the universal enveloping algebra \( \mathcal{U}(\mathfrak{p}_+) \) and the linearly independent elements \( X_1, \ldots, X_n \in \mathcal{U}(\mathfrak{p}_+) \). We now introduce a holomorphic family of elements \( E_z = \exp(z_1 X_1 + \ldots + z_n X_n) \), where \( z = (z_1, \ldots, z_n) \in \Omega \) and \( \Omega \) is an open subset of \( \mathbb{C}^n \). Suppose that there exists a cyclic vector \( \Phi_0 \in \mathcal{D} \) with \( \mathfrak{p}_- \Phi_0 = \{0\}, \quad t \Phi_0 = \Phi_0 \) and \( \Phi_0 = 1 \).
This assumption is supported by [2]. Consider now the holomorphic vectors \( \Phi_z = E_z \Phi_0 \), where \( z \in \Omega \).

The map \( T : \mathcal{H} \to \mathcal{H}_{hol}(\Omega) \) is defined by \( T(\varphi)(z) = \langle \Phi_z, \varphi \rangle \), \( \varphi \in \mathcal{H} \), \( z \in \Omega \), where \( \mathcal{H}_{hol}(\Omega) \) is the space of all \( \mathbb{C} \)-valued holomorphic functions on \( \Omega \). Consider an inner product on \( T(\mathcal{H}) \) such that \( T \) is unitary. We will to obtain an infinitesimal irreducible unitary representation \( T^*T^{-1} \) on \( T(\mathcal{H}) \) and the explicit form of the corresponding basis differential operators. Let

\[
\Phi_{\alpha\tau} = \exp \left( \alpha a^+ + \frac{\mu}{2} a^+ a + \tau W \right) \Phi_0,
\]

and \( \Psi_{\alpha\tau} = \left[ \Phi_{\alpha\tau} \right]^{-1} \Phi_{\alpha\tau} \). Then the Bargmann coherent states [12], the \( SL(2, \mathbb{R}) \) coherent states [12], and the Perelomov coherent states for \( G^J \) [4] are realized by the holomorphic vectors \( \Psi_{\alpha00} \) with \( \alpha \in \mathbb{C} \), \( \Psi_{00\tau} \) with \( \tau \in \mathbb{C} \), and \( \Psi_{\alpha\tau} \) with \( (\alpha, \tau) \in \mathbb{C} \times \mathbb{C} \), respectively.

### 2.2.2. Schrödinger-Weil representation of \( G^J \)

Let \( \mathcal{H} = L^2(\mathbb{R}) = \mathcal{H}_0 \) with the Schwartz space \( \mathcal{D} = S(\mathbb{R}) \) [2]. Let \( \pi = \pi_{SW}^m \) be the Schrödinger-Weil representation of \( G^J \). The basis differential operators of \( \tilde{\mathcal{H}}_{SW}^m(\mathfrak{g}^J) \) can be written as [2]:

\[
P = \frac{d}{dq}, \quad Q = 2i\mu q, \quad R = i\mu I, \quad F = i\mu q^2, \quad G = \frac{i}{4\mu} \frac{d^2}{dq^2}, \quad H = q \frac{d}{dq} + \frac{1}{2} I.
\]

Then \( 2K_\mu = a^2 \). In quantum mechanics, \( \hbar = (2\mu^2)^{-1} \) is the Plank constant, \( q \) is the position operator, \( p = -i\hbar d/dq \) is the momentum operator, and \( a \) and \( a^\dagger \) are the Fock annihilation and creation operators, respectively [12]. The vacuum vector \( \varphi_0 = (2\mu/\pi)^{1/4} \exp \left( -\mu |x|^2 \right) \) and the number vectors \( \varphi_n = (n!)^{-1/2} (a^\dagger)^n \varphi_0 \), where \( n \in \mathbb{N} \), form a complete orthonormal basis of analytic vectors for the Hilbert space \( \mathcal{H}_0 \). Consider the map \( T_B(\varphi) : \mathcal{H}_0 \to \mathcal{H}_{hol}(\mathbb{C}) \) defined by \( T_B(\varphi)(\alpha) = \langle \Phi_{\alpha00}, \varphi \rangle \), \( \varphi \in \mathcal{H}_0 \), \( \alpha \in \mathbb{C} \), where \( \Phi_0 = \varphi_0 \). The polynomials \( f_{Bn} = T_B(\Phi_n) \), defined by \( f_{Bn}(z) = (n!)^{-1/2} z^n \), where \( n \in \mathbb{N} \), form a complete orthonormal basis of analytic vectors in the Hilbert space \( T_B(\mathcal{H}_0) = L^2_{hol}(\mathbb{C}, \mu_B) \) with the Bargmann measure given by \( d\mu_B = \pi^{-1} \exp \left( -|z|^2 \right) d^2z \). The geometric quantization of \( \pi_{SW}^m \) and \( \pi_B = T_B \pi_{SW}^m T_B^{-1} \) is presented in [13]. Consider now the map \( T_0 : \mathcal{H}_0 \to \)}
→ ℋ_{hol}(ℂ × ℂ) defined by \( T_0(\varphi)(\alpha, w) = \langle \Phi_{\alpha, w}, \varphi \rangle \), \( \varphi \in ℋ_0 \), \( (\alpha, w) \in ℂ × ℂ \), where \( \Phi_0 = \varphi_0 \). The polynomials \( f_n = T_0(\varphi_n) \), where \( n \in \mathbb{N} \), form a complete orthonormal basis of analytic vectors for the Hilbert space \( T_0(ℋ_0) \). We have

**Proposition 2.** a) The generating function of the polynomials \( f_n \) can be written as

\[
T_B(\Phi_{\alpha, w})(z) = \exp \left( \alpha z + \frac{1}{2} w z^2 \right) = \sum_{n \geq 0} \frac{z^n}{\sqrt{n!}} f_n(\alpha, w). \tag{9}
\]

b) Any solution \( f \in ℋ_{hol}(ℂ × ℂ) \) of the equation \( (\partial^2 / \partial \alpha^2 - 2\partial / \partial w) f = 0 \) can be written as \( f = \sum_{n \geq 0} c_n f_n \), where \( c_n \in ℂ \). The map \( T_0 \) is non-surjective.

c) Let \( \pi_0 = T_0 \pi_{SW}^{-1} \). The basis differential operators can be expressed as

\[
\hat{\pi}_0(\alpha) = \frac{\partial}{\partial \alpha}, \quad \hat{\pi}_0(K_-) = \frac{\partial}{\partial w}, \quad \hat{\pi}_0(K_0) = \frac{1}{4} + \frac{\alpha}{2} \frac{\partial}{\partial \alpha} + w \frac{\partial}{\partial w}, \quad \hat{\pi}_0(a^+) = \alpha + w \frac{\partial}{\partial \alpha}, \quad \hat{\pi}_0(K_+) = \frac{1}{2} \alpha^2 + \frac{w}{2} + \alpha w \frac{\partial}{\partial w} + w^2 \frac{\partial}{\partial w}. \tag{10}
\]

The Schrödinger-Weil representation \( \pi_{SW} \), the Bargmann representation \( \pi_B \), and the coherent state representation \( \pi_0 \) of \( G^J \) are unitarily equivalent.

### 2.2.3. Discrete series representations

Let \( k > 1/2 \). Using Waldspurger’s representation theory of \( Mp(2, ℍ) \) [2, 10] and [14], we consider the infinitesimal representations \( \hat{\pi}_k \) of the positive discrete series on the Hilbert space \( ℋ_k \) characterized by the normalized cyclic vector \( \phi_0 \) and the complete orthonormal basis \( \phi_{n,n'}^{(k)} \), where \( n, n' \in \mathbb{N} \), with \( a \phi_0 = 0 \), \( K_- \phi_0 = 0 \), \( 2K_0 \phi_0 = k \phi_0 \), and \( \phi_{n,n'}^{(k)} = C_{n,n'}^{(k)}(\alpha)^n(W_+)^{n'} \phi_0 \). Here \( C_{n,n'}^{(k)} = [n!n'^!(k - 1/2)_n]^{-1/2} \) with \( (x)_n = \Gamma(n + x)/\Gamma(x) \). Consider now the map \( T_k(\varphi) : ℋ_k \to ℋ_{hol}(ℂ) \) defined by \( T_k(\varphi)(z, \zeta) = \langle \Phi_{\alpha, w}^{(k)}, \varphi \rangle \), \( \varphi \in ℋ_k \), where \( \Phi_0 = \varphi_0 \) and \( z, \zeta \in ℂ \) with \( |\zeta| < 1 \).

The polynomials \( j_{n,n'}^{(k)} = T_k(\phi_{n,n'}^{(k)}) \), where \( n, n' \in \mathbb{N} \), form a complete orthonormal basis of analytic vectors for the Hilbert space \( T_k(ℋ_k) \). Here \( j_{n,n'}^{(k)}(z, \zeta) = D_{n,n'}^{(k)} z^n \zeta^n \), where \( D_{n,n'}^{(k)} = (n!n'^!(k - 1/2)_n)^{1/2} \). Consider \( \hat{\pi}_k = T_k \hat{\pi}_k T_k^{-1} \).

The basis differential operators can be written as
\[ \hat{a}_k(a) = \frac{\partial}{\partial z}, \quad \hat{a}_k(a^\dagger) = z, \quad \hat{\sigma}_k(K_0) = \frac{k}{2} + \frac{1}{2} z \frac{\partial}{\partial z} + \zeta \frac{\partial}{\partial \zeta}, \]
\[ \hat{\sigma}_k(K_+) = \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial \zeta}, \quad \hat{\sigma}_k(K_-) = \frac{\zeta^2}{2} + (k - \frac{1}{2}) \zeta + \zeta^2 \frac{\partial}{\partial \zeta}. \]

Let \( d\mu_k = (k - 3/2)\pi^{-2} \exp(-|\zeta|^2)(1 - |\zeta|^2)^{k-1/2} \, d\zeta \, d^2 \zeta \). If \( k > 3/2 \), then \( T_k(\mathcal{H}_k) = L^2_\text{hol}(\mathbb{C} \times \mathbb{R}, \mu_k) \). There is an analytic continuation of \( \hat{r}_k \) in the limit \( k \to 3/2 \).

### 3. SQUEEZED STATES FOR THE JACOBI GROUP

#### 3.1. MATRIX ELEMENTS

We consider the "squeezed operator" \( T(\alpha, w) = D(\alpha)S(w) \) for \( G^J \), where the displacement operator \( D(\alpha) \) and the squeezed operator \( S(w) \) are unitary operators defined by [12]:
\[ D(\alpha) = \exp(\alpha a^\dagger - \alpha a) = \exp \left( -\frac{1}{2} |\alpha|^2 \right) \exp(\alpha a^\dagger) \exp(-\alpha a), \]
\[ S(w) = \exp(w K_+) \exp(\eta K_0) \exp(-\bar{w} K_-), \]
where \( \alpha \in \mathbb{C}, \; w \in \mathbb{C}, \; |w| < 1 \), and \( \eta = \ln(1 - |w|^2) \). The matrix elements of \( D(\alpha) \) are given by Schwinger’s formula in terms of Laguerre polynomials [12]. The matrix elements of \( S(w) \) for the Schrödinger-Weil representation can be written as associated Legendre functions [12]. Moreover, the matrix elements of \( S(w) \) for discrete series representations of \( G^J \) can be expressed in terms of hypergeometric polynomials [14]:
\[ \langle \psi^{(kn)}_n | S(w) | \psi^{(kn)}_{n'} \rangle = \frac{\lambda_{kn}}{\lambda_{kn'}} \left( 1 - |w|^2 \right)^{h/2} F \left(-n, s + n + 2h; s + 1; |w|^2 \right), \]
where \( n, n' \in \mathbb{N}, \; k > 1/2, \; h = (2k - 1)/4, \; \lambda_{kn} = [c \Gamma(2h + c) \Gamma(2h)]^{1/2} \) for \( c = n, n' \), and \( s = n' - n \geq 0 \).

We now introduce the "squeezed state vectors" \( T(\alpha, w)\phi \) for the Jacobi group \( G^J \), where \( \phi \in \mathcal{D} \) and \( \|\phi\| = 1 \). The standard coherent states, squeezed states, displaced number states, squeezed number states, and displaced squeezed number states [15] are realized by \( T(\alpha, 0)\phi_0, \; T(0, w)\phi_0, \; T(\alpha, 0)\phi_n, \; T(0, w)\phi_n \), and \( T(\alpha, w)\phi_n \), respectively [15]. The squeezed states for \( G^J \) can be in particular the coherent states introduced by Schrödinger [16], the squeezed states
considered by Kennard [17], and the displaced squeezed number states of Husimi [18].

We now introduce the notation
\[
\hat{A} = S(-w)D(-\alpha)AD(\alpha)S(w)
\]
and
\[
r = (1 - |w|^2)^{-1/2}.
\]
We obtain
\[
\hat{a} = (\hat{a}^\dagger)^\dagger = r (a + wa^\dagger) + \alpha I
\]
and
\[
\hat{K}_- = \hat{K}_-^\dagger = r^2 (K_- + 2wK_0 + w^2K_+) + r\alpha(a + wa^\dagger) + \frac{\alpha^2}{2}I,
\]
\[
\hat{K}_0 = r^2 \left[ wK_- + (1 + |w|^2)K_0 + wK_+ \right] + r\Re\alpha(a^\dagger + \bar{w}a) + \frac{|\alpha|^2}{2}I.
\]
If \( A = a^\dagger a K_{m}^n K_{n}^m a^\dagger \), then \( \hat{A} = \hat{a}^\dagger a \hat{K}_{m}^n \hat{K}_{n}^m \hat{a}^\dagger \). Using the preceding results, we can obtain the expectation values of any polynomial operator in infinitesimal generators of \( G^j \).

3.2. UNCERTAINTY RELATIONS

Consider the Schrödinger inequality [19] \( \sigma_{AA}\sigma_{BB} \geq \sigma_{AB}^2 + 4\langle A, B \rangle \) for the selfadjoint operators \( A \) and \( B \), where \( \sigma_{AB} = \langle AB + BA \rangle / 2 - \langle A \rangle \langle B \rangle \) and \( \sigma_{CC} = \langle C^2 \rangle - \langle C \rangle^2 \) for \( C = A, B \). Here \( \langle \rangle \) means the expectation value with respect to the state vector \( \Phi \in \mathcal{D} \). Consider now \( \Phi = T(\alpha, w)\phi_n \) or \( \Phi = T(\alpha, w)\phi_{nk}^{(k)} \). Let \( u_\pm = r^2 (1 \pm w)(1 \pm \bar{w}) \) and \( n_0 = n + 1/2 \). We obtain \( \sigma_{qq} = n_0 h u_+ \), \( \sigma_{pp} = n_0 h u_- \), and \( \sigma_{pq} = 2n_0 h r^2 w \).

We have \( \sigma_{qq}\sigma_{pp} = \sigma_{pq}^2 + h^2/4 \) for the squeezed states \( T(\alpha, w)\phi_0 \) and \( T(\alpha, w)\phi_0 \). Moreover, \( \sigma_{qq}\sigma_{pp} = h^2/4 \) for the coherent states \( T(\alpha, 0)\phi_0 \) and \( T(\alpha, 0)\phi_0 \). Evidently, \( \sqrt{\sigma_{qq}\sigma_{pp}} \geq n_0 h \), and we have squeezing in the region \( 2n_0 u_+ < 1 \), described by the open disk \( \|(2n_0 + 1)w + 2n_0\| < 1 \), \( w \in \mathbb{C} \).

3.3. MANDEL’S PARAMETER

Consider the Mandel parameter \( Q = \langle (\Delta N)^2 \rangle / \langle N \rangle - 1 \), where \( N = a^\dagger a \) is the number operator. We obtain
\[
Q(\alpha, w) = \frac{(4n_0^2 + 3)w^2 + 4n_0 |\alpha\bar{w} + \bar{\alpha}^2|^2 (1 - |w|^2)}{2n_0(1 - |w|^2) + (2 |\alpha|^2 - 1)(1 - |w|^2)^2} - 1.
\]
\((15)\)
We have $Q(\alpha, 0) = n\left(2|\alpha|^2 - 1\right)(n + |\alpha|^2)^{-1}$. Then $Q(\alpha, 0) = 0$ for $n = 0$ or $|\alpha| = 1/\sqrt{2}$. Moreover, $Q(0, w) = 0$ for $|w|^2 = \frac{1}{2(2n_0 + 1)}\left(\sqrt{16n_0^2 + 24n_0^2 - 3 - 4n_0^2} - 1\right)$. (16)

The preceding formulas are compatible with [20, 21, 15].

REFERENCES