We consider construction of some Lagrangians which contain the Riemann zeta function. The starting point in their construction is $p$-adic string theory. These Lagrangians describe some nonlocal and nonpolynomial scalar field models, where nonlocality is controlled by the operator valued Riemann zeta function. The main motivation for this research is intention to find an effective Lagrangian for adelic scalar strings.

1. INTRODUCTION

Since 1987, $p$-adicity has been found in string theory as well as in many other models of modern mathematical physics (for an early review, see e.g. [1, 2]). One of the most important results in $p$-adic string theory is construction of an effective Lagrangian for open scalar $p$-adic string tachyon [3, 4]. This simple Lagrangian describes four-point scattering amplitudes as well as all higher ones at the tree-level.

Using this Lagrangian many aspects of $p$-adic string dynamics have been investigated and compared with dynamics of ordinary strings. Mathematical study of spatially homogeneous solutions of the relevant nonlinear differential equation of motion has been of particular interest (see [5, 6] and references therein). Some possible cosmological implications of $p$-adic string theory have been also investigated [7, 8]. As a result of these developments, there have been established many similarities and analogies between $p$-adic and ordinary strings.

Adelic approach to the string scattering amplitude connects its $p$-adic and ordinary counterparts (see [1, 2] as a review). It gives rise to introduce adelic strings, which contain ordinary and $p$-adic strings as particular adelic states. Adelic generalization of quantum mechanics was also successfully formulated, and it was found a connection between adelic vacuum state of the harmonic oscillator and the Riemann zeta function [9]. Roughly speaking, adelic description is more profound than real or $p$-adic ones separately. Consequently, adelic strings are more fundamental objects than ordinary ones, described by real numbers.
The present paper can be regarded as a result of some attempts to construct an effective Lagrangian for adelic scalar string. Our approach is as follows: we start with the exact Lagrangian for the effective field of $p$-adic tachyon string, extend prime number $p$ to arbitrary natural number $n$, undertake various summations of such Lagrangians over $n$, and obtain some scalar field models with the operator valued Riemann zeta function. This zeta function controls spacetime nonlocality. These scalar field models are also interesting in their own right.

2. CONSTRUCTION OF ZETA LAGRANGIANS

The exact tree-level Lagrangian of effective scalar field $\phi$ for open $p$-adic string tachyon is

$$
L_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \left[ - \frac{1}{2} \frac{\phi}{m_p} \frac{1}{2} \frac{\partial}{\partial \phi} \frac{1}{2} \frac{\partial}{\partial \phi} \phi + \frac{1}{p+1} \phi^{p+1} \right],
$$

where $p$ is any prime number, $\Box = -\partial^2 + \nabla^2$ is the $D$-dimensional d’Alembertian.

The equation of motion for (1) is

$$
-p^{2m_p} \phi = \phi^p,
$$

and its properties have been studied by many authors (see, e.g. [5, 6] and references therein).

Prime number $p$ in (1) and (2) can be replaced by any natural number $n \geq 2$ and such expressions also make sense. Moreover, taking $p = 1 + \epsilon \to 1$ there is the limit of (1)

$$
L_0 = \frac{m_0^D}{g_0^2} \left[ - \frac{1}{2} \frac{\phi}{m_0} \frac{1}{2} \frac{\partial}{\partial \phi} \frac{1}{2} \frac{\partial}{\partial \phi} \phi + \frac{\phi^2}{2} (\ln \phi^2 - 1) \right],
$$

which corresponds to the ordinary bosonic string in the boundary string field theory [10].

Now we want to introduce a Lagrangian which incorporates all the above Lagrangians (1), with $p$ replaced by $n \in \mathbb{N}$, and $L_0$ (3). To this end, we take the sum of all Lagrangians $L_n$ in the form

$$
L = \sum_{n=0}^{\infty} C_n L_n = C_0 L_0 + \sum_{n=1}^{\infty} C_n \frac{m_n^D}{g_n^2} \frac{n^2}{n-1} \left[ - \frac{1}{2} \frac{\phi}{m_n} \frac{1}{2} \frac{\partial}{\partial \phi} \frac{1}{2} \frac{\partial}{\partial \phi} \phi + \frac{1}{n+1} \phi^{n+1} \right],
$$

whose explicit realization depends on particular choice of coefficients $C_n$, string masses $m_n$ and coupling constants $g_n$. To avoid a divergence in $1/(n-1)$ when
$n = 1$ one has to take that $C_n m_n^D g_n^2$ is proportional to $n - 1$. Here we shall consider some cases when coefficients $C_n$ are proportional to $n - 1$, while masses $m_n$ as well as coupling constants $g_n$ do not depend on $n$, i.e. $m_n = m$, $g_n = g$. Since this is an attempt towards effective Lagrangian of an adelic string it seems natural to take mass and coupling constant independent on particular $p$ or $n$. To differ this new field from a particular $p$-adic one, we use notation $\phi$ instead of $\varphi$.

2.1. THREE TYPES OF LAGRANGIANS

We are going to consider three cases, which depend on the choice of coefficients $C_n$, $n \geq 1$.

2.1.1. Case $C_n = \frac{n-1}{n^{2+h}}$

Let us first consider the case

$$C_n = \frac{n-1}{n^{2+h}},$$

(5)

where $h$ is a real number. The corresponding Lagrangian is

$$L_h = C_0 \mathcal{L}_0 + \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \sum_{n=1}^{+\infty} \frac{n^{-h}}{2^{m^2} + h} \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right]$$

(6)

and it depends on parameter $h$.

According to the famous Euler product formula, one can write

$$\sum_{n=1}^{+\infty} \frac{n^{-h}}{2^{m^2} + h} = \prod_{p} \frac{1}{1 - p^{-\frac{2}{m^2} + h}}.$$ 

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1,$$

(7)

which has analytic continuation to the entire complex $s$ plane, excluding the point $s = 1$, where it has a simple pole with residue 1. Employing definition (7), we can rewrite (6) in the form

$$L_h = C_0 \mathcal{L}_0 + \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \zeta \left( \frac{2}{m^2} + h \right) \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right].$$

(8)
Here \( \zeta\left(\frac{\Box}{2m^2} + h\right) \) acts as a pseudodifferential operator

\[
\zeta\left(\frac{\Box}{2m^2} + h\right)\phi(x) = \frac{1}{(2\pi)^D} \int e^{ikx} \zeta\left(-\frac{k^2}{2m^2} + h\right) \tilde{\phi}(k) dk, \tag{9}
\]

where \( \tilde{\phi}(k) = \int e^{-ikx} \phi(x) dx \) is the Fourier transform of \( \phi(x) \). Lagrangian \( L_0 \), with the restriction on momenta \( -k^2 = k_0^2 - \vec{k}^2 > (2 - 2h)m^2 \) and field \( |\phi| < 1 \), is analyzed in [11]. In [12], we considered Lagrangian (8) with analytic continuations of the zeta function and the power series

\[
\sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1}, \text{i.e.}
\]

\[
L_n = C_0 L_0 + \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \left( \zeta\left(\frac{\Box}{2m^2} + h\right) + \frac{n^{\mu(n)} - 1}{2m^2} \phi \right) + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right], \tag{10}
\]

where \( \mu(n) \) denotes analytic continuation.

Nonlocal dynamics of this field \( \phi \) is encoded in the pseudodifferential form of the Riemann zeta function. When the d’Alembertian is in the argument of the Riemann zeta function we say that we have zeta nonlocality. Accordingly, this \( \phi \) is a zeta nonlocal scalar field.

2.1.2. Case \( C_n = \frac{n^2 - 1}{n^2} \)

In this case Lagrangian (4) becomes

\[
L = C_0 L_0 + \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \sum_{n=1}^{+\infty} \left( \frac{n^{-2m^{n+1}}}{n-2m^{n+1}} + \frac{n^{-2m^{n}}}{n-2m^{n}} \right) \phi + \sum_{n=1}^{+\infty} \phi^{n+1} \right], \tag{11}
\]

and it yields

\[
L = C_0 L_0 + \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \left( \zeta\left(\frac{\Box}{2m^2} - 1\right) + \zeta\left(\frac{\Box}{2m^2}\right) \right) \phi + \frac{\phi^2}{1 - \phi} \right]. \tag{12}
\]

Some classical field properties of this Lagrangian are analyzed and presented in [13].

2.1.3. Case \( C_n = \mu(n) \frac{n-1}{n^2} \)

Here \( \mu(n) \) is the Möbius function, which is defined for all positive integers and has values 1, 0, –1 depending on factorization of \( n \) into prime numbers \( p \). It is defined as follows:
The corresponding Lagrangian is
\[
L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2m^2}} \phi + \int_0^\phi \mathcal{M}(\phi) d\phi \right]
\] (14)

Recall that the inverse Riemann zeta function can be defined by
\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + i\tau, \quad \sigma > 1.
\] (15)

Now (14) can be rewritten as
\[
L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \left( \frac{1}{\zeta\left( \frac{\sigma}{2m^2} \right)} \right) \phi + \int_0^\phi \mathcal{M}(\phi) d\phi \right],
\] (16)

where \( \mathcal{M}(\phi) = \sum_{n=1}^{\infty} \mu(n)\phi^n = \phi - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \ldots \).

The corresponding potential, equation of motion and mass spectrum formula, respectively, are:
\[
V_\mu(\phi) = -L_\mu (\square = 0) = \frac{m^D}{g^2} \left[ \frac{C_0}{2} \phi^2 (1 - \ln \phi^2) - \phi^2 - \int_0^\phi \mathcal{M}(\phi) d\phi \right],
\] (17)

\[
\frac{1}{\zeta\left( \frac{\sigma}{2m^2} \right)} \phi - \mathcal{M}(\phi) - \frac{C_0}{m^2} \phi - 2C_0 \phi \ln \phi = 0,
\] (18)

\[
\frac{1}{\zeta\left( \frac{M^2}{2m^2} \right)} - C_0 \frac{M^2}{m^2} + 2C_0 - 1 = 0, \quad |\phi| \ll 1,
\] (19)

where usual relativistic kinematic relation \( k^2 = -k_0^2 + \vec{k}^2 = -M^2 \) is used.

Analysis of the above expressions will be presented elsewhere.

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