We present solutions for the (constant and spectral-parameter) Yang-Baxter equations and Yang-Baxter systems, arising from algebra structures. We discuss about their applications in theoretical physics.

1. INTRODUCTION AND PRELIMINARIES

The quantum Yang-Baxter equation (QYBE) first appeared in theoretical physics and statistical mechanics. It plays a crucial role in analysis of integrable systems, in quantum and statistical mechanics, in knot theory, and also in the theory of quantum groups. On the other hand, the theory of integrable Hamiltonian systems makes great use of the solutions of the one-parameter form of the Yang-Baxter equation, since coefficients of the power series expansion of such a solution give rise to commuting integrals of motion.

Non-additive solutions of the two-parameter form of the QYBE are referred to as a colored Yang-Baxter operator. They appear in this paper, and, in this case, they are related to the solutions of the one-parameter form of the Yang-Baxter equation.

Yang-Baxter systems emerged from the study of quantum integrable systems, as generalizations of the QYBE related to nonultralocal models.

This paper presents some of the latest results on Yang-Baxter operators from algebra structures and related topics: colored Yang-Baxter operators, Yang-Baxter systems and applications in theoretical physics. In the last section, we present an enhanced version of Theorem 1 (from [7]), a physical model is chosen, and the steps to reach the Yang-Baxter equation are evidentiated. Also, some research projects are presented. We omitted the proofs, but the reader is encouraged to check the results by direct computations.
Throughout this paper $k$ is a field. All tensor products appearing in this paper are defined over $k$. For $V$ a $k$-space, we denote by $\tau: V \otimes V \to V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$, and by $I: V \to V$ the identity map of the space $V$.

We use the following notations concerning the Yang-Baxter equation.

If $R: V \otimes V \to V \otimes V$ is a $k$-linear map, then $R^{12} = R \otimes I, \ R^{23} = I \otimes R, \ R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

**Definition 1.1.** An invertible $k$-linear map $R: V \otimes V \to V \otimes V$ is called a Yang-Baxter operator if it satisfies the equation

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{1.1}$$

**Remark 1.2.** The equation (1.1) is usually called the braid equation. It is a well-known fact that the operator $R$ satisfies (1.1) if and only if $R \circ \tau$ satisfies the constant QYBE (if and only if $R \circ \tau$ satisfies the constant QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} \tag{1.2}$$

**Remark 1.3.** (i) $\tau: V \otimes V \to V \otimes V$ is an example of a Yang-Baxter operator.

(ii) An exhaustive list of invertible solutions for (1.2) in dimension 2 is given in [3] and in the appendix of [4].

(iii) Finding all Yang-Baxter operators in dimension greater than 2 is an unsolved problem.

Let $A$ be a (unitary) associative $k$-algebra, and $\alpha, \beta, \gamma \in k$. We define the $k$-linear map: $R^{A}_{\alpha, \beta, \gamma}: A \otimes A \to A \otimes A, \ R^{A}_{\alpha, \beta, \gamma}(a \otimes b) = \alpha ab + \beta b \otimes ab - \gamma a \otimes b$.

**Theorem 1.4.** (S. Dăscălescu and F. F. Nichita, [2]) Let $A$ be an associative $k$-algebra with $\dim A \geq 2$, and $\alpha, \beta, \gamma \in k$. Then $R^{A}_{\alpha, \beta, \gamma}$ is a Yang-Baxter operator if and only if one of the following holds:

(i) $\alpha = \gamma \neq 0; \beta \neq 0$;

(ii) $\beta = \gamma \neq 0; \alpha \neq 0$;

(iii) $\alpha = \beta = 0, \gamma \neq 0$.

If so, we have $(R^{A}_{\alpha, \beta, \gamma})^{-1} = R^{A}_{\frac{1}{\beta}, \frac{1}{\alpha}, \frac{1}{\gamma}}$ in cases (i) and (ii), and $(R^{A}_{0, 0, \gamma})^{-1} = R^{A}_{0, 0, \frac{1}{\gamma}}$ in case (iii).

We now present the matrix form of the operator obtained in the case (i) of the previous theorem, $R = R^{A}_{\alpha, \beta, \alpha}: A \otimes A \to A \otimes A, \ R(a \otimes b) = \alpha ab \otimes 1 +$
Some results on the Yang-Baxter equations

1. Let $A$ be an associative algebra with $\dim A \geq 2$. We consider the algebra $A = \frac{k[X]}{(X^2 - mX - n)}$, where $m, n$ are scalars. Then $A$ has the basis $\{1, x\}$, where $x$ is the image of $X$ in the factor ring. We consider the basis $\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$ of $A \otimes A$ and represent the operator $R$ in this basis:

\[
R(1 \otimes 1) = \beta 1 \otimes 1 \\
R(1 \otimes x) = (\beta - \alpha) 1 \otimes x + \alpha x \otimes 1 \\
R(x \otimes 1) = \beta 1 \otimes x \\
R(x \otimes x) = (\alpha + \beta)n 1 \otimes 1 + \beta m 1 \otimes x + \alpha mx \otimes 1 - \alpha x \otimes x
\]

In matrix form, this operator reads:

\[
\begin{pmatrix}
\beta & 0 & 0 & 0 \\
0 & \beta - \alpha & \alpha & 0 \\
0 & 0 & \beta & 0 \\
(\alpha + \beta)n & \beta m & \alpha m & -\alpha
\end{pmatrix}
\]  

(1.3)

2. THE TWO-PARAMETER FORM OF THE QYBE

Formally, a colored Yang-Baxter operator is defined as a function $R : X \times X \rightarrow \text{End}_k(V \otimes V)$, where $X$ is a set and $V$ is a finite dimensional vector space over a field $k$. Thus, for any $u, v \in X$, $R(u, v) : V \otimes V \rightarrow V \otimes V$ is a linear operator. We consider three operators acting on a triple tensor product $V \otimes V \otimes V$, $R^{12}(u, v) = R(u, v) \otimes I$, $R^{23}(v, w) = I \otimes R(v, w)$, and similarly $R^{13}(u, w)$ as an operator that acts non-trivially on the first and third factor in $V \otimes V \otimes V$.

If $R$ satisfies the two-parameter form of the QYBE:

\[
R^{12}(u, v)R^{13}(u, w)R^{23}(v, w) = R^{23}(v, w)R^{13}(u, w)R^{12}(u, v)
\]

$\forall u, v, w \in X$, then it is called a colored Yang-Baxter operator.

**Theorem 2.1.** (F. F. Nichita and D. Parashar, [6]) Let $A$ be an associative $k$-algebra with $\dim A \geq 2$, and $X \subseteq k$. Then, for any two parameters $p, q \in k$, the function $R : X \times X \rightarrow \text{End}_k(A \otimes A)$ defined by

\[
R(u, v)(a \otimes b) = p(u - v) 1 \otimes ab + q(u - v)ab \otimes 1 - (pu - qv)b \otimes a,
\]

satisfies the colored QYBE (2.4).
We now consider the algebra $A = \frac{k[X]}{(X^2 - \sigma)}$, where $\sigma$ is a scalar. Then $A$ has the basis $\{1, x\}$, where $x$ is the image of $X$ in the factor ring. We consider the basis $\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$ of $A \otimes A$.

In the matrix form the operator (2.5) reads:

\[
R(u, v) = \begin{pmatrix}
qu - pv & 0 & 0 & \sigma(q + p)(u - v) \\
0 & p(u - v) & (q - p)v & 0 \\
0 & (q - p)u & q(u - v) & 0 \\
0 & 0 & 0 & qv - pu
\end{pmatrix}
\] (2.6)

3. YANG-BAXTER SYSTEMS

It is convenient to describe the Yang-Baxter systems in terms of the Yang-Baxter commutators.

Let $V$, $V'$, $V''$ be finite dimensional vector spaces over the field $k$, and let $R: V \otimes V' \to V \otimes V', S: V \otimes V'' \to V \otimes V''$ and $T: V' \otimes V'' \to V' \otimes V''$ be three linear maps. The Yang-Baxter commutator is a map $[R, S, T]: V \otimes V' \otimes V'' \to V \otimes V' \otimes V''$ defined by

\[
[R, S, T] := R_{12}S_{13}T_{23} - T_{23}S_{13}R_{12}.
\] (3.7)

Note that $[R, R, R] = 0$ is just a short-hand notation for writing the constant QYBE (1.2).

A system of linear maps $W: V \otimes V \to V \otimes V$, $Z: V' \otimes V' \to V' \otimes V'$, $X: V \otimes V' \to V \otimes V'$, is called a WXZ–system if the following conditions hold:

\[
[W, W, W] = 0 \quad [Z, Z, Z] = 0 \quad [W, X, X] = 0 \quad [X, X, Z] = 0
\] (3.8)

It was observed that WXZ-systems with invertible $W$, $X$ and $Z$ can be used to construct dually paired bialgebras of the FRT type leading to quantum doubles. The above is one type of a constant Yang-Baxter system that has recently been studied in [6] and also shown to be closely related to entwining structures [1].

**Theorem 3.1.** (F. F. Nichita and D. Parashar, [6]) Let $A$ be a $k$-algebra, and $\lambda, \mu \in k$. The following is a WXZ-system: $W: A \otimes A \to A \otimes A$, $W(a \otimes b) = ab \otimes 1 + \lambda 1 \otimes ab - b \otimes a$, $Z: A \otimes A \to A \otimes A$, $Z(a \otimes b) = \mu ab \otimes 1 + +1 \otimes ab - b \otimes a$, $X: A \otimes A \to A \otimes A$, $X(a \otimes b) = ab \otimes 1 + 1 \otimes ab - b \otimes a$.  

4. APPLICATIONS AND RESEARCH PROJECTS

Using the techniques from above we now present an enhanced version of Theorem 1 (from [7]). It remains a research project to extend that theorem for Yang-Baxter systems and spectral-parameter dependent Yang-Baxter equations.

**Theorem 4.1.** Let $V = W \oplus kc$ be a $k$-space, and $f, g : V \otimes V \rightarrow V \otimes V$ $k$-linear maps such that $f, g = 0$ on $V \otimes c + c \otimes V$. Then, $R : V \otimes V \rightarrow V \otimes V$, $R(v \otimes w) = f(v \otimes w) \otimes c + c \otimes g(v \otimes w)$ is a solution for QYBE (1.2).

Next, a physical model is chosen and the steps to reach the Yang-Baxter equation are evidenced [5]. This is the one-dimensional Bose gas consisting of $n$ sort particles with the Dirac delta-function two particle potential. The field operators $\psi_a(x)$, $\psi_b^\dagger(y)$ satisfy the canonical commutation relations, $a, b = 1, 2, \ldots, n$

$$[\psi_a(x), \psi_b^\dagger(y)] = \delta_{ab} \delta(x - y), \quad [\psi_a(x), \psi_f(y)] = 0. \quad (4.9)$$

The Hamiltonian field is

$$H = \int dx(\partial_x \psi_a^\dagger(x) \partial_x \psi_a(x) + c\psi_a^\dagger(x)\psi_b^\dagger(x)\psi_b(x)\psi_a(x)) \quad (4.10)$$

The eigenfunctions for a fixed number of particle $M$ and a given momentum distribution $\{\lambda_j\}$ it is written:

$$\Psi_{M} = \int d^M x \Psi(1, \ldots, M | \lambda_1, \ldots, \lambda_M) \prod_{i=1}^{M} \psi_{a_i}^\dagger(x_i)|0\rangle \quad (4.11)$$

with

$$\Psi(\sigma | \{\lambda_j\}) = \sum_{\sigma \in S_M} A_\sigma(\{a_j\}, \{\lambda_n\}) \exp \left( i \sum_{m=1}^{M} \lambda_{\sigma m} \chi_m \right) \quad (4.12)$$

Such a form of the Hamiltonian eigenfunctions is known as the **coordinate Bethe Ansatz**. The conditions of the wave function continuity and its appropriate derivative jumps on the hyperplanes $x_j = x_{j+1}$ (the sewing conditions) define the coefficients $A_\sigma(\{a_j\}, \{\lambda_k\})$. The coefficients $A_\sigma$ and $A_{\sigma'}$ with $\sigma' = \sigma \sigma$ where $\sigma_j$ is the transposition of the indices $j, j+1$, are related by the two particle $S$-matrix:

$$A_{\sigma'} = S(\lambda_j - \lambda_{j+1}) A_\sigma, \quad (4.13)$$

where for the model chosen 4.10 $S(\lambda) = (\lambda + ic \mathcal{P})/(\lambda - ic)$, and $\mathcal{P}$ is the permutation operator in $C^n \otimes C^n$. 
The periodicity condition for the system on finite interval \((0, L)\) results in the **Bethe equations** for the set of \(M\) momenta \(\lambda_j\)

\[
\exp(i\lambda_j L) = -\prod_k S_{jk}(\lambda_j - \lambda_k),
\]

where in the ordered product \(k = j + 1, \ldots, M - 1, M, 1, \ldots, j - 1\). Hence the meaning of the RHS is the scattering matrix of the \(j\)-th particle on the other \((M - 1)\) particles. This would be just a phase factor for the scalar particles \((S(\lambda) = (\lambda + ic)/(\lambda - ic))\), but for the \(n\) component case one has to diagonalize the complicated scattering matrix to arrive to a system of scalar equations.

The consistency condition of this system is the **Yang-Baxter equation** (YBE) for the \(S\)-matrix \(S(\lambda)\):

\[
S_{jk}(\lambda_j - \lambda_k)S_{jl}(\lambda_j - \lambda_l)S_{kl}(\lambda_k - \lambda_l) = S_{kl}(\lambda_k - \lambda_l)S_{jl}(\lambda_j - \lambda_l)S_{jk}(\lambda_j - \lambda_k).
\]

An interesting line of research is thus to investigate whether or not the particular type of solution of the YBE expressed in the previous sections do satisfy the YBE of the physical system described above (or of other physical systems).

REFERENCES