FROM MASSIVE GRAVITY TO DARK MATTER DENSITY

G. SCHARF
Institut für Theoretische Physik, Universität Zürich, Winterthurerstr. 190, CH-8057 Zürich, Switzerland; e-mail: scharf@physik.unizh.ch

Received August 30, 2008

Massive gravity previously constructed as the spin-2 quantum gauge theory is studied in the classical limit. The vector graviton field \(v\) which does not decouple in the limit of vanishing graviton mass gives rise to a modification of general relativity. The modified Schwarzschild solution contains a contribution from the \(v\)-field which can be interpreted as the dark matter mass density. We calculate the corresponding density profile in the simplest spherically symmetric geometry.

1. INTRODUCTION

Massive gravity is a controversial subject because it is very difficult to construct it starting from a classical lagrangian theory. On the other hand, there exists a powerful method to find the quantum theory directly: the derivation of gauge couplings from a cohomological formulation of gauge invariance in terms of asymptotic free fields. This method works for massless and massive gauge theories equally well. For spin-1 theories this was demonstrated in the monograph [1], where the massless spin-2 case is also treated. The massive spin-2 theory was first investigated in [2]. The most elegant way to obtain the theory is by assuming the gauge invariance condition for all chronological products in the form of the descent equations [3]. These give the total interaction Lagrangian including ghost couplings and the necessary coupling to a vector-graviton field. By comparison with the couplings derived from expansion of the Einstein Lagrangian in powers of the coupling constant we observe that we really have a quantum theory of gravity. The massive spin-2 theory corresponds to the Einstein Lagrangian with a negative cosmological constant. So this is not directly related to the dark energy.

By analogy with spin-1 one would expect that if the graviton is massive, a gravitational Higgs field would be necessary to save gauge invariance in higher orders. This would be a natural candidate for dark matter. However, the theory is gauge invariant in second order without any additional field [2], and in third order the anomalies checked so far also cancel. A gravitational Higgs seems not to exist.
There is a second option in massive gravity. A massive spin-2 particle has 5 degrees of freedom in contrast to 2 of the massless graviton. It turns out that there are even 6 physical modes in massive gravity [4]. In gauge theory there is always some freedom to choose the physical states. In order to have a smooth massless limit \( m \to 0 \) of the massive theory, one is forced to choose the 6 physical modes as follows: two of them are the transversal modes of the massless graviton, the remaining four are generated by a vector-graviton field \( v^\lambda \), which is required in the massive theory to have it gauge invariant. The surprising fact is that for \( m \to 0 \) this vector-graviton field \( v^\lambda \) does not decouple from the symmetric tensor field \( h^{\mu\nu} \) which corresponds to the classical \( g^{\mu\nu} \) of Einstein. Consequently, the massless limit of massive gravity is not general relativity; there is a modification due to the (now massless) vector graviton field \( v^\lambda \). We will show that this modification gives rise to an additional force which looks as if it comes from a dark matter density.

2. MASSIVE QUANTUM GRAVITY

The basic free asymptotic fields of massive gravity are the symmetric tensor field \( h^{\mu\nu}(x) \) with arbitrary trace, the fermionic ghost \( u^\mu(x) \) and anti-ghost \( u_\mu(x) \) fields and the vector-graviton field \( v^\lambda(x) \). They all satisfy the Klein-Gordon equation

\[
(\square + m^2) h^{\mu\nu} = 0 = (\square + m^2) u^\mu = (\square + m^2) u_\mu = (\square + m^2) v^\lambda
\]

and are quantized as follows [2, 3]

\[
[h^{\alpha\beta}(x), h^{\mu\nu}(y)] = -ih^{\alpha\beta\mu\nu} D_m(x - y)
\]

with

\[
h^{\alpha\beta\mu\nu} = \frac{1}{2}(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu}),
\]

\[
\{u^\mu(x), u_\nu(y)\} = i\eta^{\mu\nu} D_m(x - y)
\]

\[
[v^\mu(x), v_\nu(y)] = i\frac{1}{2}\eta^{\mu\nu} D_m(x - y),
\]

and zero otherwise. Here, \( D_m(x) \) is the Jordan-Pauli distribution with mass \( m \) and \( \eta^{\mu\nu} = \text{diag}(1,-1,-1,-1) \) the Minkowski tensor.

The gauge structure on these fields is defined through a nilpotent gauge charge operator \( Q \) satisfying

\[
Q^2 = 0, \quad Q\Omega = 0
\]

where \( \Omega \) is the Fock vacuum and
The vector-graviton field $v^\lambda$ is necessary for nilpotency of $Q$. The vacuum $\Omega$ exists on Minkowski background only, therefore, apart from simplicity this background is preferred for physical reasons.

The coupling $T(x)$ between these fields follows from the gauge invariance condition [2]

$$d_Q T(x) = i \partial_\alpha T^\alpha(x)$$

(2.10)

where $T$ and $T^\alpha$ are normally ordered polynomials with ghost number 0 and 1, respectively. In addition we may require the descent equations

$$\partial_\alpha T^\alpha = [Q, T^\alpha] = i \partial_\beta T^{\alpha\beta}, \quad [Q, T^{\alpha\beta}] = i \partial_\gamma T^{\alpha\beta\gamma}$$

(2.11)

where the new $T$'s are antisymmetric in the Lorentz indices. The essentially unique coupling derived from (2.10–11) is given by [3]

$$T = h^{\alpha\beta} \partial_\alpha h \partial_\beta h - 2h^{\alpha\beta} \partial_\alpha h^{\mu\nu} \partial_\beta h_{\mu\nu} - 4h^{\alpha\beta} \partial_\alpha h^{\mu\nu} \partial_\beta h_{\mu\nu} - 2h^{\alpha\beta} \partial_\alpha h_{\mu\nu} \partial_\beta h^{\mu\nu}$$

$$+ 4h^{\alpha\beta} \partial_\alpha h_{\mu\nu} \partial_\beta h^{\mu\nu} + 4\partial_\alpha h^{\mu\nu} \partial_\beta h_{\mu\nu} - 4\partial_\alpha u^{\mu\nu} \partial_\beta h^{\mu\nu} - 4m u^{\alpha\beta} \partial_\alpha u^{\gamma\delta} - m^2 \left( \frac{4}{3} h_{\mu\nu} h^{\mu\nu} h^{\alpha\beta} - h^{\mu\beta} h^{\alpha\nu} + \frac{1}{6} h^3 \right).$$

(2.12)

Here $h = h^{\mu\nu}$ is the trace and a coupling constant is arbitrary. The quartic couplings follow from second order gauge invariance and so on. We consider the limit $m \to 0$ in the following. The massless limit of massive gravity is certainly a possible alternative to general relativity. The new physics comes from the surviving coupling term of the vector-graviton

$$T_v = 4h^{\alpha\beta} \partial_\alpha v^\beta \partial_\beta v^\lambda.$$  

(2.13)

To be able to do non-perturbative calculations in massive gravity we look for the classical theory corresponding to the coupling (2.12). It was shown in [1] Sect. 5.5 that the pure graviton couplings $h h h h$ correspond to the Einstein-Hilbert Lagrangian

$$L_{\text{EH}} = \frac{2}{\kappa^2} \sqrt{-g} R$$

(2.14)

in the following sense. We write the metric tensor as
and expand $L_{EH}$ in powers of $\kappa$. Then the quadratic terms $O(\kappa^0)$ give the free theory, the cubic terms $O(\kappa^1)$ agree with the pure graviton coupling terms in (2.12) up to a factor 4 and so on. To obtain $T_v$ (2.13) we must add

$$L_v = \sqrt{-g} g_{\alpha\beta} \partial_\alpha v_\lambda \partial_\beta v^\lambda$$

(2.16)

to $L_{EH}$. One may be tempted to write covariant derivatives $\nabla_\alpha$ instead of partial derivatives in order to get a true scalar under general coordinate transformations. But this would produce quartic couplings containing $v_\lambda$ and such terms are absent in the quantum theory ([2], eq. (4.12)). For the same reason the Lorentz index $\lambda$ in $v_\lambda$ is raised and lowered with the Minkowski tensor $\eta_{\mu\nu}$, but all other indices with $g_{\mu\nu}$. Both together means that the vector graviton field $v_\lambda$ should be considered as four scalar fields in the classical theory, whereas in the quantum theory it is a genuine 4-vector field.

The total classical Lagrangian we have to study is now given by

$$L_{tot} = L_g + L_M + L_v$$

(2.17)

where

$$L_g = \frac{c^3}{16\pi G} \sqrt{-g} R$$

(2.18)

we have introduced the velocity of light and Newton’s constant. $L_M$ is given by the energy-momentum tensor $t_{\mu\nu}$ of ordinary matter as usual. Calculating first the variational derivative with respect to $g_{\mu\nu}$ we get the following modified Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{8\pi G}{c^4} t_{\mu\nu} = \frac{16\pi G}{c^4} \left( -\partial_\mu v_\lambda \partial_\nu v^\lambda + \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha v_\lambda \partial_\beta v^\lambda \right).$$

(2.19)

Secondly the variation with respect to $v_\lambda$ gives

$$\partial_\alpha \left( \sqrt{-g} g^{\alpha\beta} \partial_\beta v^\lambda \right) = 0.$$  

(2.20)

After multiplication with $1/\sqrt{-g}$ this is the Laplace-Beltrami or rather the wave equation in the metric $g^{\alpha\beta}$. The coupled system (2.19) (2.20) is the modification of general relativity which we want to study.

### 3. MODIFIED SCHWARZSCHILD SOLUTION

We wish to construct a static spherical symmetric solution of the modified field equations. Following the convention of [5] we write the metric as
\[ ds^2 = e^\nu c^2 dt^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) - e^\lambda dr^2 \]  
(3.1)

where \( \nu \) and \( \lambda \) are functions of \( r \) only. We take the coordinates \( x^0 = ct, x^1 = r, \) \( x^2 = \vartheta, x^3 = \varphi \) such that

\[ g_{00} = e^\nu, \quad g_{11} = -e^\lambda, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \vartheta \]  
(3.2)

and zero otherwise. The components with upper indices are the inverse of this.

At first we solve the Laplace-Beltrami equation (2.20) which can be written in the form

\[ \frac{\partial}{\partial r} \left( e^{(\nu-\lambda)/2} r^2 \frac{\partial v_\kappa}{\partial r} \right) = e^{(\nu+\lambda)/2} L^2 v_\kappa, \]  
(3.3)

where \( L^2 \) is the quantum mechanical angular momentum operator squared. Consequently the angular dependence of \( v_\kappa \) is given by spherical harmonics \( Y_l^m (\vartheta, \varphi) \):

\[ v_\kappa (r, \vartheta, \varphi) = v_\kappa (r) Y_l^m (\vartheta, \varphi) \]  
(3.4)

and the radial function \( v_\kappa (r) \) satisfies the following radial equation

\[ \frac{\partial}{\partial r} \left( e^{(\nu-\lambda)/2} r^2 \frac{\partial v_\kappa (r)}{\partial r} \right) = l(l+1) e^{(\nu+\lambda)/2} v_\kappa (r). \]  
(3.5)

For what follows we consider the case \( l = 0 \) where we get the simple result

\[ v'_\kappa (r) = \frac{C_\kappa}{r^2} e^{(\lambda-\nu)/2}, \]  
(3.6)

here \( C_\kappa, \kappa = 0, 1, 2, 3 \) are constants of integration.

Using standard techniques [5] the modified radial Einstein equations reads

\[ G_0^0 = e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{8\pi G}{c^3} (q(r)c + w_0 (r)) \]  
(3.7)

\[ -G_1^1 = e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{8\pi G}{c^3} \left( \frac{p_1 (r)}{c} + w_1 (r) \right) \]  
(3.8)

\[ -G_2^2 = e^{-\lambda} \left( \frac{\nu'}{2} + \frac{\nu^2}{4} - \frac{\lambda'}{2r} + \frac{\lambda}{2r} \right) - \frac{1}{r^2} = \frac{8\pi G}{c^3} \left( \frac{p_2 (r)}{c} + w_2 (r) \right). \]  
(3.9)

\[ -G_3^3 = e^{-\lambda} \left( \frac{\nu'}{2} + \frac{\nu^2}{4} - \frac{\lambda'}{2r} + \frac{\lambda}{2r} \right) = \frac{8\pi G}{c^3} \left( \frac{p_3 (r)}{c} + w_3 (r) \right), \]  
(3.10)

where the energy-momentum tensor of ordinary matter is assumed in the diagonal form \( t^\alpha_\alpha = \text{diag}(qc^2, -p_1, -p_2, -p_3) \) and \( w_0, w_1, w_2, w_3 \) are the contributions from the \( v \)-field in (2.19). If we assume \( p_2 = p_3 \), we must have
\( w_2 = w_3 \) and this is only possible for \( l = 0 \). Then \( v_k \) depends on \( r \) only. We restrict ourselves to this case in the following, i.e. \( p_j = p \).

From (3.6) we now obtain the following contributions of the \( v \)-field

\[
\varrho^{11} \partial_{11} v \partial_{11} v = \frac{1}{r^4} e^{-\nu} (C_1^2 + C_2^2 + C_3^2 - C_0^2) = \frac{C}{r^4} e^{-\nu} \tag{3.11}
\]

and zero otherwise, so that

\[
w_0 = \frac{C}{r^4} e^{-\nu} = w_2 = w_3, \quad w_1 = -w_0. \tag{3.12}
\]

We see in (3.7) that the mass density \( q(r) \) gets enlarged by the quantity \( w_0(r)/c \) which we shall call dark density. We omit the notion “matter” because this energy density comes from the vector-graviton field, it is a relic of the massive graviton.

Next by suitable combination we simplify the equations. Adding (3.7) to (3.8) we get

\[
\frac{\nu''}{\nu} = \frac{\lambda''}{\nu} + \frac{\nu'}{\nu} = \frac{8\pi G}{c^2} \left( q + \frac{p}{c^2} \right). \tag{3.13}
\]

From (3.7–10) we find

\[
\frac{p'}{c^2} = -\frac{\nu'}{2} \left( q + \frac{p}{c^2} + \frac{2}{c} w_0 \right). \tag{3.14}
\]

For a first orientation we want to solve the equations neglecting the ordinary matter, i.e. \( q = 0 = p \). Then we find from (3.13) \( \lambda' + \nu' = 0 \) or

\[
v(r) = D_1 - \lambda(r) \tag{3.15}
\]

where \( D_1 \) is a constant of integration. Using this is (3.12) we have

\[
w_0 = \frac{D_2}{r^4} e^\lambda. \tag{3.16}
\]

Multiplying (3.7) by \( r^2 \) and introducing

\[
y = re^{-\lambda} \tag{3.17}
\]

we write the equation in the form

\[
y' = 1 - 8\pi \frac{D_3}{ry} \quad \text{where} \quad D_3 = \frac{GD_2}{c^3}. \tag{3.18}
\]

Although this equation looks rather simple, we did not succeed in expressing the solution by known special functions, but a numerical solution can be easily obtained. The dark density \( w_0 \) (3.12) is given in terms of \( y \) (3.17) by
\[ w_0 = \frac{D_2}{r^3 y(r)}. \] (3.19)

For large \( r \) we find the following power series expansion
\[ y = r - a_1 + \frac{a_2}{r} + \frac{a_3}{r^2} + \ldots \] (3.20)
where
\[ a_2 = 8\pi D_3, \quad a_3 = 4\pi D_3 a_1, \ldots \] (3.21)

\( a_i \) is related to the total mass \( M \) by the Schwarzschild relation
\[ a_1 = 2GM \frac{1}{c^2} = r_s. \] (3.22)

That means the tail of the dark density profile (3.19) can be approximated by
\[ w_0 = \frac{D_2}{r} \frac{1}{r^3 - a_1 r^2 + a_2 r + a_3}, \quad r \geq r_c. \] (3.23)

On the other hand Navarro, Frenk and White [6] have found the following profile
\[ \varphi(r) = \frac{\varphi_s}{r \left( 1 + \frac{r}{r_s} \right)^2} \]
from N-body simulations, where \( r_s \) is the so-called scaling radius and \( \varphi_s \) the scaling density. Obviously, this profile is adjusted to the intermediate and inner part of the halo where visible matter cannot be neglected. To make contact with this result we must calculate a solution with ordinary matter. This will be investigated elsewhere.

REFERENCES