We comment on the definition and use of gauge invariance in less usual circumstances, exemplifying with mechanics on noncommutative spaces.

1. INTRODUCTION

It is by now well-known that naive gauge invariance is broken when vector fields are defined over a noncommutative (NC) space (see e.g. [1]). A way out is to render the gauge field non-Abelian through the introduction of the star-product [2]. We present here an alternative view of the phenomenon.

In usual mechanics the symplectic two-form $\omega$ is usually taken to be block-diagonal from the start,

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i.$$  \hspace{1cm} (1)

$q_i, p_i$, $i = 1, \ldots, n$ are the phase-space variables of the system under consideration, to be denoted collectively by $x_a$, $a = 1, \ldots, 2n$ in the following.

Interesting consequences arise if one allows for a more general, even if still constant, symplectic form

$$\omega = \sum_{a,b=1}^{2n} \omega_{ab}^x dx_a \wedge dx_b.$$  \hspace{1cm} (2)

Classically, (2) generates extended Poisson brackets

$$\{x_a, x_b\} = \Theta_{ab},$$  \hspace{1cm} (3)

where $\Theta$ is the inverse of $\omega$, $\Theta_{ab} = (\omega^{-1})_{ab}$. Quantum mechanically, one replaces the Poisson brackets with commutators. After reviewing the formalism, we demonstrate gauge non-invariance, propose a more general framework, and apply it via diagonalization in noncommutative mechanics.
2. NONCOMMUTATIVE DYNAMICS

We will work in (2+1)-dimensions, for simplicity in notation. A convenient starting point is the classical action

\[ S = \int dt \left( \frac{1}{2} \omega_{ab} x_a \dot{x}_b - H(x) \right), \quad x, \dot{x} = q, \dot{q}, p, \dot{p}, \quad x_1, x_2, x_3, x_4 = q_1, q_2, p_1, p_2, \]

which engenders the equations of motion,

\[ \dot{x}_a = \{ x_a, H \} = \Theta_{ab} \frac{\partial H}{\partial x_b}, \quad \Theta_{ab} = (\omega^{-1})_{ab}, \quad a, b = 1, 2, 3, 4. \]

Above, \{A, B\} = \Theta_{ab} \frac{\partial A}{\partial x_b} \frac{\partial B}{\partial x_a}; in particular, \{x_a, x_b\} = 0, \{x_a, p_b\} = \Theta_{ab}, \{p_a, p_b\} = 0, \{p_a, x_b\} = \epsilon_{ab}. If we choose the symplectic form to be

\[ \Theta = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & \sigma \\ 0 & -1 & -\sigma & 0 \end{pmatrix}, \quad \omega = \frac{1}{1-\theta\sigma} \begin{pmatrix} 0 & -\sigma & 1 & 0 \\ \sigma & 0 & 0 & 1 \\ -1 & 0 & 0 & -\theta \\ 0 & -1 & -\theta & 0 \end{pmatrix}, \]

the nonzero Poisson brackets are \{q_i, p_j\} = \delta_{ij}, \{q_1, q_2\} = \theta, \{p_1, p_2\} = \sigma \) and the phase-space equations of motion become

\[ \dot{q}_i = \frac{\partial H}{\partial p_i} + \theta \epsilon_{ij} \frac{\partial H}{\partial q_j}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \sigma \epsilon_{ij} \frac{\partial H}{\partial p_j}, \quad \epsilon_{12} = -\epsilon_{21} = 1. \]

If \theta = \sigma = 0, Eqs. (7) are the usual Hamilton equations. Eliminating the momenta from (7) one gets, for Hamiltonians of the form \( H = p^2/2 + V(q) \), the following coordinate equations of motion:

\[ \ddot{q}_i = -(1-\theta\sigma) \frac{\partial V}{\partial q_i} + \sigma \epsilon_{ij} \dot{q}_j + \theta \epsilon_{ij} \frac{d}{dt} \frac{\partial V}{\partial q_j}, \quad i = 1, 2. \]

If \theta \neq 0, Eqs. (8) are not in general derivable from a Lagrangian [4].

To quantize the theory, it is enough to promote the phase-space variables \( x_a \) to operators, and consider the extended Heisenberg algebra (with \( h = 1 \), for simplicity)

\[ [\hat{q}_i, \hat{p}_j] = i \delta_{ij}, \quad [\hat{q}_1, \hat{q}_2] = i \theta, \quad [\hat{p}_1, \hat{p}_2] = i \sigma, \quad [\hat{p}_1, \epsilon_{ij} \hat{q}_j] = 0. \]

The dynamics is simply given by

\[ \frac{d\hat{x}_a}{dt} = -i [\hat{x}_a, H] = \Theta_{ab} \frac{\partial H}{\partial \hat{x}_b}, \]

in the Heisenberg representation. A generic operator \( O \) satisfies the equation of motion \( i\dot{O} = [H, O] = \frac{\partial H}{\partial \hat{x}_a} [\hat{x}_a, \hat{x}_b] \frac{\partial O}{\partial \hat{x}_b}. \)
A Schrödinger formulation can be found also \cite{4}. Given that $[\hat{q}_1, \hat{p}_2] = 0$, one can for instance define the wave function $\psi(q_1, p_2)$. The commutation relations (9) enforce then the following action of the other operators:

$$\hat{p}_1 \psi(q_1, p_2) = -i(\partial_{q_1} - \sigma \partial_{p_2}) \psi(q_1, p_2), \quad \hat{q}_2 \psi(q_1, p_2) = i(\partial_{p_2} - \sigma \partial_{q_1}) \psi(q_1, p_2)$$

(11)

Similar arguments lead to other representations. If $\theta \neq 0$, no coordinate wavefunction $\psi(q_1, q_2)$ exists.

3. BREAKING OF USUAL GAUGE INVARIANCE

Consider now a Hamiltonian with minimally coupled gauge field $A_i,$

$$H = \frac{1}{2} \sum_{i=1,2} \left[ p_i - A_i(q_i) \right]^2.$$ \hspace{1cm} (12)

Then Eqs. (7) imply (using the notation $\frac{\partial A_j}{\partial q_i} = \partial_i A_j$ )

$$\dot{q}_i = (p_i - A_i) \left[ \delta_{ij} - \theta \epsilon_{ij} \partial_j A_i \right], \quad \dot{p}_j = (p_j - A_j) \left[ \partial_j A_j + \sigma \epsilon_{ij} \right]. \hspace{1cm} (13)$$

Assuming $\frac{\partial A_i}{\partial t} = 0,$ (13) can be rewritten as

$$\dot{p}_i = A_i + \frac{1}{\Delta} \frac{d}{dt} (q_i + \theta \epsilon_{ij} A_j), \quad i = 1, 2, \hspace{1cm} (15)$$

where $\Delta = 1 + \theta F_{12} + \theta^2 \{ A_1, A_2 \}_{q\neq 2},$ with $F_{12} = \partial_1 A_2 - \partial_2 A_1, \{ A_1, A_2 \}_{q\neq 2} = \partial_1 \partial_2 A_1 - \partial_2 \partial_1 A_2.$ Using (15) in (14), and assuming (for simplicity) $\partial_1 A_1 = \partial_2 A_2 = 0,$ one gets

$$\dot{q}_1 = \left( 1 - \theta \partial_2 A_1 \right) \left[ - A_1 + (\partial_1 A_2 + \sigma) \frac{\dot{q}_2}{1 + \theta \partial_2 A_1} \right], \hspace{1cm} (16)$$

$$\dot{q}_2 = \left( 1 + \theta \partial_1 A_2 \right) \left[ - A_2 + (\partial_2 A_1 + \sigma) \frac{\dot{q}_1}{1 + \theta \partial_1 A_2} \right]. \hspace{1cm} (17)$$

Let us consider the case of a constant magnetic field, $B = F_{12} = \partial_1 A_2 - \partial_2 A_1.$ This can be obtained in different gauges. A striking feature of the equations (16, 17) is that they are not gauge invariant, unless $\theta = 0.$ For instance, in the symmetric gauge, $A_1 = -q_2 B/2, A_2 = q_1 B/2,$ one has

$$\dot{q}_1 = \dot{q}_2 \left( \sigma + B + \theta B^2 / 4 \right), \quad \dot{q}_2 = -\dot{q}_1 \left( \sigma + B + \theta B^2 / 4 \right) \hspace{1cm} (18)$$

whereas in the gauge $A_1 = 0, A_2 = q_1 B$ one gets...
\[
\hat{q}_1 = \hat{q}_2 \frac{(\sigma + B)}{1 + \theta B} \quad \hat{q}_2 = -\hat{q}_1 (\sigma + B)(1 + \theta B),
\]

which is not even derivable from a Lagrangian. Whereas \(\sigma\) just adds to \(B\), mimicking thus a magnetic field, \(\theta\) even breaks gauge invariance! A reformulation of the concept appears necessary in this context. It is presented in the next subsection, directly at the quantum level.

### 4. OPERATORIAL FORMULATION OF GAUGE INVARIANCE

Let us first consider the stationary Schrödinger equation in usual quantum mechanics, where one has \(\hat{q}_i, \hat{p}_j = i \delta_{ij}\). In presence of a connection, it reads

\[
\left[ \sum_k \left( i \frac{\partial}{\partial q_k} + A_k(q_i) \right)^2 + V(q_i) \right] \Psi(q_i) = E \Psi(q_i).
\]

Eq. (20) can however be put into operatorial form, namely

\[
\left[ \sum_k (\hat{q}_k + A_k(\hat{q}_i))^2 + V(\hat{q}_i) \right] \Psi(q_i) = E \Psi(q_i),
\]

and the gauge transformation (21) becomes a Hilbert space transformation

\[
|q_i \rangle \rightarrow e^{i\Lambda(\hat{\phi}_i)} |q_i \rangle \quad A_k(\hat{q}_i) \rightarrow A_k(\hat{q}_i) + \frac{\partial}{\partial q_k} \Lambda(q_i).
\]

That this is the case can be easily seen by using the identity

\[
e^A e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \ldots.
\]

In usual QM only the first commutator matters. In general, under a gauge transformation for which

\[
|\Psi \rangle \rightarrow e^{i\Lambda} |\Psi \rangle,
\]

an operator \(O\) shifts as follows:

\[
O \rightarrow O + i[\Lambda, O] + \frac{i^2}{2!} [\Lambda, [\Lambda, O]] + \frac{i^3}{3!} [\Lambda, [\Lambda, [\Lambda, O]]] + \ldots
\]

This provides a general formulation for gauge invariance, which includes the usual commutative case, but also NC mechanics. Explicitely, to ensure gauge invariance for generic commutation relations it is sufficient to suitably generalize
the transformations (21). For the NC mechanics we discussed, one must find $\Lambda$ and a transformation law for $A_i, A_i \rightarrow A_i + \delta A_i$, which ensure
\[
(i\hat{p}_k + A_i(q_i) + \delta A_i)e^{i\Lambda} = e^{i\Lambda}(i\hat{p}_k + A_i(q_i)).
\] (26)

Eq. (25) becomes essential; it now defines $\delta A_i$. Given that the infinite sum over multiple commutators is hard to evaluate when coordinates do not commute, we employ the diagonalization procedure in the following example.

5. DIAGONALIZATION

Consider one of the simplest ways to trade (9) for a usual Heisenberg algebra [3], by defining the canonical phase space coordinates:
\[
Q_2 = q_2, \quad P_i = p_i, \quad Q_i = q_i + \frac{\theta p_2}{1 - \theta \sigma}, \quad P_2 = \frac{p_2 + \sigma q_1}{1 - \theta \sigma}.
\] (27)

In terms of those coordinates usual gauge invariance works. Thus minimal coupling amounts to the substitution $H(P, Q) \rightarrow H(P - A, Q)$ (it does not matter for our purposes that the Hamiltonian in terms of $Q$ and $P$ could be quite complicated), and the gauge transformation is given by Eq. (21). If we shift back to $q, p$ coordinates however,
\[
q_2 = Q_2, \quad p_1 = P_1, \quad q_1 = Q_1 - \theta p_2, \quad p_2 = P_2 - \sigma Q_1,
\] (28)

we see that the minimal substitution becomes
\[
p_1 \rightarrow p_1 - A_i(q, p), \quad q_1 \rightarrow q_1 + B_1,
\] (29)

where
\[
A_i(q, p) = A_i(Q_i, Q_2) = A_i\left(\frac{q_i + \theta p_2}{1 - \theta \sigma}, q_2\right), \quad B_i(q, p) = 0A_2.
\] (30)

Obviously $\Lambda$ in (21) depends now on the momenta too,
\[
\Lambda(Q) \rightarrow \Lambda(q, p) = \Lambda(Q_1, Q_2) = \Lambda\left(\frac{q_i + \theta p_2}{1 - \theta \sigma}, q_2\right),
\] (31)

and the transformation law for $\Lambda_i(q, p)$ involves
\[
\partial A_i = \partial_{Q_i} \Lambda(Q_1, Q_2) = \partial_{q_i} \Lambda - \sigma \partial_{p_2} \Lambda, \quad \delta A_i = \partial_{q_2} \Lambda.
\] (32)

In conclusion, gauge invariance of the Schrödinger equation can be preserved at the price of introducing fields minimally coupled to coordinates also, and $p$-dependence in the function $\Lambda$ entering the gauge transformation.
6. DIAGONALIZATION WITH A FREE PARAMETER

Finally, let us show that the full matrix (6) can be diagonalized without fixing a continuous parameter. This shows that a form of gauge invariance persists also for the NC algebra, even in the case when $\sigma \theta \neq 0$ (the case $\sigma \theta = 0$ is discussed in [4]). One such (gauge) transformation depending on a continuous parameter is, for $Q, P$, canonical,

\[
q_1 = Q_1 (1 - \alpha \sigma \theta) - \theta p_2, \quad q_2 = Q_2,
\]
\[
p_1 = P_1 + \alpha \sigma Q_2, \quad p_2 = P_2 - (1 - \alpha) \sigma Q_1.
\]

For any value of the real parameter $\alpha$, the phase space coordinates $q_{1,2}, p_{1,2}$ in (33) obey the algebra (9). When $\alpha = 0$ one reobtains (28).

REFERENCES