We discuss the formulation of mechanics with inhomogeneous noncommutativity at the classical and quantum level.

Mechanics with noncommuting coordinates has recently been a subject of considerable interest [1]. Classical extensions to inhomogeneous noncommutativity have also been discussed, in particular for the case in which dimensional reduction takes place [2]. Our purpose in this work is mainly to extend to variable commutators a path integral formulation for noncommutative mechanics, initially proposed in [3] for constant commutators. As customary in the field, we work in 2+1 dimensions, for simplicity in notation. Coordinates are denoted by \( q_1 \) and \( q_2 \), momenta by \( p_1 \) and \( p_2 \), and both types of phase space coordinates by \( x_a, x_{1,2,3,4} = q_1, q_2, p_1, p_2 \). Assume the Poisson brackets of the theory to be

\[
\{q_1, q_2\} = \Theta(q, p), \quad \{p_1, p_2\} = \sigma(q, p), \quad \{q_i, p_j\} = \delta_{ij},
\]

or equivalently \( \{x_a, x_b\} = \Theta_{ab}, \Theta_{ab} = (\omega^{-1})_{ab} \), where

\[
\Theta = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & \sigma \\ 0 & -1 & -\sigma & 0 \end{pmatrix}, \quad \text{i.e.} \quad \omega = \frac{1}{1-\theta \sigma} \begin{pmatrix} 0 - \sigma & 1 & 0 \\ \sigma & 0 & 0 & 1 \\ -1 & 0 & 0 - \theta \\ 0 & -1 & \theta & 0 \end{pmatrix}.
\] (2)

The form of \( \sigma(q, p) \) and \( \theta(q, p) \) above is restricted only by the Jacobi identities

\[
\partial_{p_2} \sigma - \partial_{q_1} \sigma = 0, \quad \partial_{q_1} \theta - \sigma \partial_{p_2} \theta = 0, \quad (3)
\]
\[
\partial_{p_1} \sigma + \Theta_{q_2} \sigma = 0, \quad \partial_{q_2} \theta + \partial_{p_1} \theta = 0. \quad (4)
\]
The simplest approach to the dynamics engendered by the brackets (1) is the variational one: we write down the action which generates equations of motion of the type

\[ \dot{x}_a = \Theta_{ab} \frac{\partial H}{\partial x_b}, \]

i.e. \( \dot{x}_a = \{x_a, H\} \) but with \( \{x_a, x_b\} = \Theta_{ab} \).

For constant \( \omega \) (or \( \Theta \)) this action is

\[ S = \int dt \left( \frac{1}{2} \omega_{ab} x_a \dot{x}_b - H(x) \right). \]

The quantum theory is described by the Hamiltonian \( H \) and the commutation relations

\[ [x_a, x_b] = i\Theta_{ab}, \quad x_{1,2,3,4} = q_1, q_2, p_1, p_2. \]

For constant \( \Theta \) the relevant path integral is

\[ Z = \int \prod_{k=1}^4 Dx_k e^{iS} = \int \prod_{k=1}^4 Dx_k e^{i \int dt \left( \frac{1}{2} \omega_{ab} x_a \dot{x}_b - H(x) \right)}. \]

To prove that (8) enforces the commutation relations (7) one only needs to know that \( Z \) represents a transition amplitude between two states of a given Hilbert space, and that time-ordering of operators is enforced, as usual, by the path integral, \( \int D\Phi e^{\frac{i}{\hbar} S_{\text{path}}(\Phi)} \) [3].

It is possible to infer the path integral formulation for inhomogeneous \( \Theta(x) \) provided one starts from the symplectic form, i.e. from the action

\[ S = \int_0^T dt \left[ A_A(x) \dot{x}_a - H(x) \right], \]

with

\[ \partial_a A_b(x) - \partial_b A_a(x) = \omega_{ab}(x) \equiv \Theta_{ab}^{-1}, \]

which leads to the equations of motion (5) for nonconstant \( \Theta \) as well. The path integral will be taken to be the continuum limit of the discretized partition function

\[ Z \approx \int \prod_{n=0,1,2,\ldots,N} I_{N+1} x_n^{(n)} \prod_{n=0,1,2,\ldots,N} e^{i \int_{x_n^{(n)}}^{x_{n+1}^{(n)}} \left[ \dot{A}_a(x) \dot{x}_a - H(x) \right] dx + \frac{i}{\hbar} \int_{x_n^{(n)}}^{x_{n+1}^{(n)}} \left[ \dot{A}_a(x) \dot{x}_a - H(x) \right] dx + \frac{i}{\hbar} \int_{x_n^{(n)}}^{x_{n+1}^{(n)}} \left[ \dot{A}_a(x) \dot{x}_a - H(x) \right] dx}, \]

where \( \varepsilon \) is the time increment, \( \varepsilon = T/N \), and \( x_n^{(n)} \) is the value of the phase space variable \( x_a \) at time \( t_0 + n\varepsilon \), \( n = 0, 1, 2, \ldots, N \). Let us consider the expectation value of \( \frac{\partial \hat{O}}{\partial \hat{x}_a^{(n)}} \), where \( \hat{O}(\hat{x}) \) is an operator depending on the \( \hat{x}_a \)'s. Integrating by parts under the path integral (11), one gets
\[
\left\langle \frac{\partial \hat{O}}{\partial \vec{x}_k^{(n)}} \right\rangle = -i \left\langle T \left\{ \hat{O}(\hat{x}_i), \frac{\partial \tilde{S}}{\partial \vec{x}_k^{(n)}} \right\} \right\rangle.
\] (12)

\(T\{,\}\) represents the time-ordering of operators, which means \((n)\)-ordering in the discrete case. \(\tilde{S}\) is the discretized form of the action sitting in the exponent of (11), and \(\frac{\partial \tilde{S}}{\partial \vec{x}_k^{(n)}} = \omega_{q_j}(x)\left( x_j^{(n+1)} - x_j^{(n-1)} \right) - e \frac{\partial \hat{H}}{\partial \vec{x}_k^{(n)}}\). Choosing \(\hat{O} = \hat{x}_i\), converting the \((n)\)-ordering into a commutator, and taking then the continuum limit \(e \to 0\), (12) becomes

\[
\sum_j \omega_{q_j}(x)\left[ \hat{x}_j, \hat{x}_k \right] = i \delta_k,
\] (13)

which implies \(\left[ \hat{x}_j, \hat{x}_j \right] = i \Theta(x) = i \omega^{-1}\), i.e. the commutation relations (7).

To derive the Heisenberg equations of motion as well, we choose \(\hat{O}\) proportional to the identity operator. Then, one gets \(\frac{\partial \tilde{S}}{\partial \vec{x}_k^{(n)}} = 0\), leading to

\[
\omega_{q_j} \frac{d}{dt} \hat{x}_j = \frac{\partial \hat{H}}{\partial \vec{x}_j}.
\]

Solving for \(\frac{d}{dt} \hat{x}_j\), one obtains

\[
\frac{d}{dt} \hat{x}_j = \Theta(x) \frac{\partial \hat{H}}{\partial \vec{x}_j} = -i \left[ \hat{x}_j, \hat{H} \right],
\] (14)

which are the extended Heisenberg equations of motion (the quantum form of (5), with \(\{,\}_{PB} \to -i\{,\}\)).

We derived the commutation relations and the operator equations of motion from the path integral (11), for inhomogeneous noncommutativity. One notes that the time ordering appearing in the previous derivation allowed us to ignore at a first glance the ordering of the operators appearing simultaneously in \(\tilde{S}\). We now switch to the derivation of the path integral from the operatorial formulation. Here ordering problems will become relevant.

Since \(\left[ \hat{p}_j, \hat{q}_j \right] = 0\), we can consider the basis spanned by the set of eigenvectors of \(\hat{q}_j\) and \(\hat{p}_j\), \(\{|q_j,p_j\}\), or alternatively by \(\{|q_j,p_l\}\).

In order to calculate the transition amplitude

\[
A = \left\langle q_1, p_2 | e^{-i\hat{H}t} | q_1^{(0)}, p_2^{(0)} \right\rangle,
\] (15)

between two states with prescribed position along the first axis of coordinates, and well defined momentum along the second axis, we know from the constant
noncommutativity case [3] that it is sufficient to evaluate \( \langle q_1, p_2 | q_2, p_1 \rangle \), which in that case was
\[
\langle q_1, p_2 | q_2, p_1 \rangle = \frac{1}{2} \exp \left( \frac{i}{1 - \theta \sigma} (q_1 p_2 - q_2 p_1 + \theta p_1 p_2 - \sigma q_1 q_2) \right). \tag{16}
\]

Consider first the case in which \( \theta = 0 \). Then we can represent
\[
[p_1, p_2] = i \sigma (q_1, q_2), \quad [q_1, p_j] = i \delta_{ij}
\]
by
\[
\hat{q}_1 | q_1, p_j \rangle = q_1 | q_1, p_j \rangle, \quad \hat{p}_1 | q_1, p_j \rangle = -i (\hat{\sigma}_{q_2} + A_1 | q_1, p_j \rangle)
\]
provided \( \hat{\sigma}_{q_2} A_2 - \hat{\sigma}_{p_2} A_1 = \sigma (q_1, q_2) \). However, we will need later on \( \langle q_1, p_2 | q_2, p_1 \rangle \) to obtain the expression for the path integral. In the \( | q_2, p_1 \rangle \) basis the nondiagonal operators can be represented by
\[
\hat{q}_1 | q_2, p_1 \rangle = i \hat{\sigma}_{p_2} | q_2, p_1 \rangle, \quad \hat{p}_1 | q_2, p_1 \rangle = -i (\hat{\sigma}_{q_2} + \hat{\sigma}_{p_2}) | q_2, p_1 \rangle.
\tag{17}
\]

Using the notation \( \langle q_1, p_2 | q_2, p_1 \rangle \equiv f = e^{-i \alpha} \), one obtains the following linear partial differential equations for \( f \)
\[
\begin{align*}
q_1 f &= i \theta f, \\
p_2 f &= -i (\hat{\sigma}_{q_2} + \hat{\sigma}_{p_1}) f.
\end{align*} \tag{18}
\]

Integrability is ensured by the Jacobi identities, and in the end one obtains
\[
\alpha = q_1 p_2 - q_2 p_1 - \int_{q_1}^{q_2} \sigma (q, q_2)
\tag{19}
\]
where an additional arbitrary function \( \phi (q_1, p_2) \) was dropped. It is a standard thing now to construct the path integral, and it is easily seen that Eqs. (10, 11) are correctly reproduced when \( \theta = 0 \): The gauge field connection emerges in the path integral under the expected form \( (\int_x \Theta (x)) \hat{x} \). If \( \theta \neq 0 \) but \( \sigma = 0 \) one can proceed similarly; one simply dualizes \( q \) and \( p \), in fact.

If however \( F \theta \neq 0 \), the derivation becomes more complicated, and ordering prescriptions become important too; one prefers to retreat to particular forms for \( \theta \) and \( \sigma \). There is however an interesting situation which can still be treated for generic \( \Theta \). The Jacobi identities (3, 4) imply that the phase space variables enter \( \theta \) and \( \sigma \) in pairs: if they depend on \( q_1 \), they must also depend on \( p_2 \), and vice versa. This is a fortunate situation, since the members of a pair commute with each other, hence no ordering problems will appear. We consider the case in which \( \Theta \) depends on only one pair, say \( (q_2, p_1) \), i.e.
\[
[\hat{q}_1, \hat{q}_2] = i \theta (\hat{q}_2, \hat{p}_1), \quad [\hat{p}_1, \hat{p}_2] = i \sigma (\hat{q}_2, \hat{p}_1), \quad [\hat{q}_1, \hat{p}_2] = i \delta_{ij}.
\tag{20}
\]

Then \( \hat{q}_1, \hat{p}_2 \) can be represented via
\[
\hat{q}_1 | q_2, p_1 \rangle = i (\hat{\sigma}_{p_1} + \hat{\sigma}_{q_2}) | q_2, p_1 \rangle, \quad \hat{p}_2 | q_2, p_1 \rangle = -i (\hat{\sigma}_{q_2} + \hat{\sigma}_{p_1}) | q_2, p_1 \rangle.
\tag{21}
\]
\((\hat{q}_1, \hat{p}_2) = 0 \) is ensured by the Jacobi identity.) This representation leads to the following differential equations for \( \langle q_1, p_2 | q_2, p_1 \rangle \equiv f = e^{-i \alpha} : 
\]
\[ \partial_{p_1} \alpha + \theta \partial_{q_2} \alpha = q_1, \quad \partial_{q_2} \alpha + \sigma \partial_{p_1} \alpha = -p_2. \] \tag{22}

The Jacobi identity (4) ensures the integrability condition, and permits to put the solution either in the form

\[ \alpha = \int_{p_1} \frac{q_1 + \theta p_2}{1 - \partial \sigma}, \] \tag{23}

with an arbitrary additive function \( \phi_1(q_1, q_2, p_2) \) dropped, or in the form

\[ \alpha = -\int_{q_2} \frac{p_2 - \sigma q_1}{1 - \theta \sigma}, \] \tag{24}

with an arbitrary function \( \phi_2(q_1, p_1, p_2) \) dropped out. In either case, the path integral (11, 10) is reconstructed.

REFERENCES
