ONE METHOD FOR CONSTRUCTION OF INVERSE ORTHOGONAL MATRICES

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An analytical procedure to obtain parametrisation of complex inverse orthogonal matrices starting from complex inverse orthogonal conference matrices is provided. These matrices, which depend on nonzero complex parameters, generalize the complex Hadamard matrices and have applications in spin models and digital signal processing. When the free complex parameters take values on the unit circle they transform into complex Hadamard matrices.

1. INTRODUCTION

In a seminal paper [1] Sylvester defined a general class of orthogonal matrices named inverse orthogonal matrices and provided the first examples of what are nowadays called (complex) Hadamard matrices. A particular class of inverse orthogonal matrices $A = (a_{ij})$ are those matrices whose inverse is given by $A^{-1} = (1/a_{ij})^t = (1/a_{ji})$, where $t$ means transpose, and their entries $a_{ij}$ satisfy the relation

$$AA^{-1} = nI_n$$

(1)

where $I_n$ is the $n$-dimensional unit matrix. When the entries $a_{ij}$ take values on the unit circle $A^{-1}$ coincides with the Hermitian conjugate $A^*$ of $A$, and in this case (1) is the definition of complex Hadamard matrices. Complex Hadamard matrices have applications in quantum information theory, several branches of combinatorics, digital signal processing, etc.

Complex orthogonal matrices unexpectedly appeared in the description of topological invariants of knots and links, see e.g. Ref. [2]. They were called two-weight spin models being related to symmetric statistical Potts models. These matrices have been generalized to two-weight spin models, also called


generalized spin models, by removing the symmetry condition [3]. After that these models have been extended to four-weight spin models, see [4]. These matrices are also known as type II matrices, \( W \), with non-zero entries which satisfy the relations

\[
\sum_{k=1}^{n} W_{ik} W_{jk} = n \delta_{ij}, \quad i, j = 1, 2, \ldots, n
\]  

(2)

see Ref. [5]. It is easily seen that the relations (1) and (2) are equivalent. In the case of spin models the above relation is supplemented with two other relations which take into account the star-triangle relations, which impose further constraints on pairs of \( W \) matrices. Particular cases of type II matrices also appeared as a generalized Hadamard transform for processing multiphase or multilevel signals, see [6] and [7], which includes the classical Fourier, Walsh-Hadamard and Reverse Jacket transforms. They are defined for \( 2^n \times 2^n \)-dimensional matrices and depend on a \( p^{th} \) root of unity and/or one complex non-zero parameter. The aim of the paper is to provide an analytic method for the construction of inverse orthogonal matrices. It is well known, see [5], that a complete solution was given only for dimensions, \( n \leq 5 \), and here we give a few examples for dimensions \( n \geq 8 \).

2. INVERSE ORTHOGONAL MATRICES

Our method for construction of inverse orthogonal matrices uses as starting point an analog of the complex \( n \times n \) conference matrices \( C_n \), which are matrices with \( a_{ii} = 0, i = 1, \ldots, n \) and \( |a_{ij}| = 1, \ i \neq j \) that satisfy

\[
C_n C_n^* = (n-1)I_n
\]  

(3)

where \( C_n^* \) is the Hermitian conjugate of \( C_n \). Conference matrices \( C_n \) are important because by construction the matrix

\[
H_{2n} = \begin{pmatrix}
C_n + I_n & C_n^* - I_n \\
C_n - I_n & -C_n^* - I_n
\end{pmatrix}
\]  

(4)

is a complex Hadamard matrix, as one can easily verify.

To extend the above construction to complex orthogonal conference matrices we have to prescribe a recipe for a proper treating of the zero entries on the main diagonal. Thus the complex inverse orthogonal conference matrices are defined by a similar relation to the relation (3), with \( a_{ij}, \ i \neq j \), complex non-zero numbers, and \( a_{ii} = 0 \). Our formula for the inverse is given by
Method for construction of inverse orthogonal matrices

\[ C^{-1} = (1 / (C_n + I_n) - I_n)^t \]  
(5)

where \(1/A\) is the matrix whose elements are \(1/a_{ij}\). We remark that the above relation makes sense since all the entries of \(C_n + I_n\) are complex non-null numbers. In fact one can generalize the above inverse for an arbitrary complex matrix, \(M\), with arbitrary located zeros, the entries of the inverse matrix \(M^{-1}\) being given by

\[ M^{-1}_{ij} = \begin{cases} 
1/M_{ji} & \text{for } M_{ij} \neq 0 \\
0 & \text{for } M_{ij} = 0 
\end{cases} \]  
(6)

In the case of complex orthogonal matrices the formula (4) takes the form

\[ O_{2n} = \begin{pmatrix} C_n + I_n & C_n^{-1} - I_n \\
C_n^{-1} - I_n & -C_n - I_n \end{pmatrix} \]  
(7)

To see how the method works we start with the simplest example, i.e. we consider the case \(n = 2\), when the orthogonal conference matrix has the form

\[ C_n = \begin{pmatrix} 0 & a \\
a & 0 \end{pmatrix} \]  
(8)

with \(a \in \mathbb{C}^*\) an arbitrary non-zero number. According to (5), or (6), the inverse is given by

\[ C_n^{-1} = \begin{pmatrix} 0 & 1/a \\
1/a & 0 \end{pmatrix} \]  
(9)

and one simply verifies that \(C_2C_2^{-1} = I_2\). By using the relation (4) one gets a complex inverse orthogonal matrix which depends on a single complex parameter, \(a\).

\[ D_4 = \begin{pmatrix} 1 & a & -1 & 1/a \\
1/a & 1 & -1 & a \\
-1 & a & 1 & -1/a \\
a & -1 & -1/a & -1 \end{pmatrix} \]  
(10)

Multiplying at right and at left by the diagonal matrices \(d_1 = \begin{pmatrix} 1, 1/a, -1, a \end{pmatrix}\), and \(d_2 = \begin{pmatrix} 1, 1/a, -1, 1/a \end{pmatrix}\), generated by the inverse of the first row, respectively the first column, one finds, modulo a transposition, the jacket form of \(D_4\) as
where $b = 1/a^2$. Jacket matrices have entries on the first row and column equal to 1, and on the last row and column $\pm 1$. When $b = e^{it}$ with $t^2 = -1$ and $t$ a real number one finds a matrix which depends on a continuous parameter, $t \in \mathbb{R}$, matrix that was found by Hadamard, [8].

Since there are no inverse complex orthogonal conference matrices for $n = 3$, we consider the case $n = 4$ and start with

$$
C_4 = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & -a & a \\
1 & a & 0 & -a \\
1 & -a & a & 0
\end{pmatrix}
$$

where $a \neq 0$ is an arbitrary complex number. It was obtained from the complex conference matrix given in [9], by changing the phase into the complex number $a$. To see what is the general procedure for obtaining inverse orthogonal matrices we multiply (12) at right and respectively at left by the diagonal matrices

$$d_1 = (a_1, a_2, a_3, a_4) \quad \text{and} \quad d_2 = (1, b_1, b_2, b_3),$$

where $a_i$ and $b_i$ are non-zero complex numbers to obtain a more general form of matrix $C_4$. By using formula (7) one gets

$$O_8 = \begin{pmatrix}
1 & a_2 & a_3 & a_4 & -1 & 1/a_1b_1 & 1/a_2b_2 & 1/a_3b_3 \\
a_1b_1 & -1 & -aa_3b_1 & aa_4b_1 & 1/a_2 & -1 & 1/aa_2b_2 & -1/aa_3b_3 \\
a_2b_2 & aa_3b_2 & 1 & -aa_4b_1 & 1/a_3 & -1/aa_3b_1 & -1 & 1/aa_3b_3 \\
a_3b_3 & -aa_2b_2 & aa_4b_1 & 1 & 1/a_4 & -1/aa_4b_2 & -1 & 1/aa_4b_3 \\
-1 & -a_2 & a_3 & a_4 & 1 & 1/a_4 & 1/aa_4b_2 & 1/aa_4b_3 \\
a_1b_1 & 1 & -aa_3b_1 & aa_4b_1 & 1/a_2 & -1 & 1/aa_2b_2 & -1 \\
a_2b_2 & aa_3b_2 & -1 & -aa_4b_1 & 1/a_3 & 1/aa_3b_1 & -1 & 1/aa_3b_3 \\
(a_3b_3 & -aa_2b_2 & aa_4b_1 & 1 & -1/a_4 & -1/aa_4b_2 & -1 & 1/aa_4b_3
\end{pmatrix}
$$

Multiplying $O_8$ at right and, respectively, at left by diagonal matrices generated by the inverse of the first row, respectively, of the first column, one finds after the substitution

$$a_1 \rightarrow 1/w, \quad a_2 \rightarrow 1/xb_1, \quad a_3 \rightarrow yb_2, \quad a_4 \rightarrow 1/zb_3$$
the final form of the matrix $O_8$, up to a few permutations necessary to put it in a more symmetric form

$$D_8 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & wx & -1 & aw & x/a & -aw & -wx & -x/a \\
1 & aw & -y/a & -aw & -1 & wy & -wy & y/a \\
1 & -aw & z/a & wz & -z/a & aw & -wz & -1 \\
1 & -aw & -z/a & -wz & z/a & aw & wz & -1 \\
1 & -wx & -1 & aw & -x/a & -aw & wx & x/a \\
1 & aw & y/a & -aw & -1 & -wy & wy & -y/a \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1
\end{pmatrix} \quad (15)$$

matrix which depends on five complex arbitrary non-zero parameters. When $a$, $w$, $x$, $y$, $z$ take values on the 5-dimensional torus $T^5$, i.e. values on the unit circle, $D_8$ gets a complex Hadamard matrix. If in (15) one makes the identification $x = y = a$ one finds, in the terminology used in [7], a jacket matrix which depends on three arbitrary non-zero parameters

$$d_8 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -aw & z/a & wz & -z/a & aw & -wz & -1 \\
1 & -aw & -z/a & -wz & z/a & aw & wz & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1
\end{pmatrix} \quad (16)$$

The matrix $d_8$ is the generalization of the jacket matrix $K_4(i)$ from [7] which depends only on $\pm 1$ and $\pm i$. It is easily seen that the above construction can be generalised such that the following proposition holds:

**Proposition 1.** Let us suppose that the $n \times n$ complex inverse orthogonal conference matrix $C_n$ has the standard form, i.e. all the entries on the first row and the first column are equal to unity, excepting that on the main diagonal which is zero. If such a $C_n$ depends on $p$ complex parameters, then the complex inverse orthogonal matrix $D_{2n}$ obtained by using formula (7) depends on $n + p$ complex parameters.

Similarly, by starting with the $n = 5$ conference matrix
where $a = e^{2\pi i/3}$, and $b \in \mathbb{C}^*$, one finds by using the above procedure the matrix

$$
C_5 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & b & ab & a^2b \\
1 & b & 0 & a^2b & ab \\
1 & ab & a^2b & 0 & b \\
1 & a^2 & ab & b & 0 
\end{pmatrix}
$$

(17)

which depends on six complex parameters: $b$, $y_1$, $y_2$, $y_3$, $y_4$, $y_5$. When these parameters take values on the six-dimensional torus $T^6$ one gets a complex Hadamard matrix, and if the following relations $y_2 = a^2b$, $y_3 = -ab$, $y_4 = b$ hold, one gets a jacket matrix depending on three independent parameters $b$, $y_1$, and $y_5$. When the remaining parameters are taken equal to unity one finds a particular case of a jacket matrix, e.g.

$$
D_{10} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
y_1y_2 & y_1/ab & aby_1 & -1 & -y_1y_2 & a^2by_1 & y_2/b & by_1 & y_2/a^2b \\
y_2 & y_2/a^2b & a^2y_1 & y_3/b & -y_1y_3 & aby_1 & -1 & y_1y_3 & y_3/ab \\
1 & -aby_1 & -1 & y_1y_4 & y_4/ab & -y_1y_4 & by_1 & y_4/ab & a^2by_1 & y_4/b \\
a^2by_1 & y_3/b & by_1 & y_5/a^2b & -y_1y_5 & y_5/ab & aby_1 & -1 & y_5/ab & aby_1 & -1 \\
1 & a^2by_1 & -y_2/b & by_1 & -y_3/a^2b & y_3y_5 & -y_2/a^2b & aby_1 & -1 & y_3/ab & -y_3/ab & aby_1 & -1 \\
1 & -y_1y_2 & y_2/ab & aby_1 & -1 & y_1y_2 & a^2by_1 & -y_2/b & by_1 & y_2/a^2b & -y_2/a^2b \\
b_1 & y_3/a^2b & a^2by_1 & -y_3/b & y_3y_5 & -ab_1 & -1 & -y_1y_3 & -y_3/ab & -y_1y_3 & -y_3/ab \\
1 & aby_1 & -1 & -y_1y_4 & -y_4/ab & y_1y_4 & by_1 & -y_4/a^2b & a^2by_1 & -y_4/b & -y_4/b \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{pmatrix}
$$

(18)

which depends on six complex parameters: $b$, $y_1$, $y_2$, $y_3$, $y_4$, $y_5$. When these parameters take values on the six-dimensional torus $T^6$ one gets a complex Hadamard matrix, and if the following relations $y_2 = a^2b$, $y_3 = -ab$, $y_4 = b$ hold, one gets a jacket matrix depending on three independent parameters $b$, $y_1$, and $y_5$. When the remaining parameters are taken equal to unity one finds a particular case of a jacket matrix, e.g.

$$
d_{10} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
a^2 & a & a & -1 & -a^2 & a^2 & 1 & 1 \\
1 & 1 & -a^2 & a^2 & -a & a & -1 & -a & -1 \\
1 & a & -1 & 1 & a^2 & -1 & 1 & a & a^2 & 1 \\
1 & a^2 & 1 & 1 & a & -1 & 1 & a^2 & a & -1 \\
1 & a^2 & -1 & 1 & -a & -1 & 1 & a^2 & a & -1 \\
1 & -a^2 & -a & a & -1 & a^2 & a^2 & -a^2 & 1 & -1 \\
1 & 1 & a^2 & a^2 & -a & a & -1 & a & a & 1 \\
1 & a & -1 & -1 & a^2 & 1 & 1 & -a & a^2 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{pmatrix}
$$

(19)

where $a = e^{2\pi i/3}$. 
When \( n = 6 \) one starts with the matrix

\[
C_6 = \frac{1}{\sqrt{6}} \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -a & -a & a & a \\
1 & -a & 0 & a & -a/b & a/b \\
1 & -a & a & 0 & a/b & -a/b \\
1 & a & -ab & ab & 0 & -a \\
1 & a & ab & -ab & -a & 0
\end{pmatrix}
\]

found in [9], by changing the continuous parameters taking values on the unit circle in arbitrary non-zero complex numbers \( a \) and \( b \). By applying the same technique one finds a complex inverse orthogonal matrix, \( D_{12} \), which depends on eight parameters, and so on.

3. CONCLUSION

In this paper we proposed a procedure to find parametrizations of complex inverse orthogonal matrices. We mention that our approach provides new complex Hadamard matrices, when the complex parameters are restricted to take values on the corresponding torus. Even if the Hadamard matrices obtained from \( D_8 \) and \( D_{10} \) depends on five, and, respectively, six arbitrary phases they are not equivalent to the matrices which can be obtained by using our method proposed in section 4 of paper [9]; see in this respect the paper [10].

The complex orthogonal conference matrices \( C_4 \) and \( C_5 \) are uniquely determined by the corresponding complex conference matrices, so the number of complex parameters entering \( D_8 \) and \( D_{10} \) is the maximal one which can be obtained by using our approach. The problem is not yet decided for the \( C_6 \) matrix, and this problem will be treated elsewhere.

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