QUANTUM OSCILLATIONS AND THE ELECTRONIC TRANSPORT PROPERTIES IN MULTICHAIN NANORINGS*

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We consider a system of multichain nanorings in static electric and magnetic field. The magnetic field induces characteristic phase changes. These phase shifts produce interference phenomena in the case of nanosystems for which the coherence length is larger than the sample dimension. We obtain energy solutions that are dependent on the number of sites \( N_{\alpha} \) characterizing a chain, of phase on the phase \( \phi_{\alpha} \) and on the applied voltage. We found rich oscillations structures exhibited by the magnetic flux and we established the transmission probability. This proceeds by applying Landauer conductance formulae which opens the way to study electronic transport properties.

Key words: multichain nanorings, quantum oscillations, transmission probability.

1. INTRODUCTION

Quantum transport property has been studied during the last years through artificially fabricated nanostructures [1–6]. Many interesting quantum effects can be also found in coupled nanostructures where the electronic transport is affected by the phenomena of quantum interference [7]. The application of a magnetic field, which is often used to probe the properties of nanodevices, induces characteristic changes in the phase coherence of the electronic wavefunctions which give rise to particular interference effects for the electronic transport [8]. For a closed path, the electron wave function experiences the phase difference \(-2\pi\Phi/\Phi_0\), as discussed before in connection with Aharonov-Bohm boundary condition [9]. The magnetic flux is denoted by \( \Phi \), while \( \Phi_0 = \hbar c/e \) denotes the flux quantum. These phase shifts produce interference phenomena in the case of nanosystems for which the coherence length is larger than the sample dimension. In this paper, we study the electron transport properties of multichain structure with leads attached at its ends. In the multichannel structure an initial wave splits

up into complementary waves $\psi_1, \psi_2, \ldots, \psi_N$, where $N$ is the total number of chains involved. These waves propagate independently in every chain and are finally recombined at the outgoing lead. In a multichain system the phase changes are not the same for different paths of propagation, so that they can lead to particular interference phenomena accompanied by much more complicated conductance oscillations than in the ordinary two-channel Aharonov-Bohm ring [10]. Each phase shift is caused by both the electronic momentum and the magnetic flux, so that momentum and chain length variations as well as variations in the distribution of the magnetic fluxes can modify the interference patterns [7]. The oscillation patterns and their periodicities are also very sensitive to the partitioning of the flux among the areas enclosed by the paths. We established the transmission probability [11, 12], by applying Landauer conductance formulae which opens the way us to study electronic transport properties. We emphasize that numerical studies of oscillation patterns, plots included, deserve further attention. This concerns the conductance, but the underlying transmission coefficient will also be considered. In order to make the reading selfconsistent basic equations written down previously [7] will be shortly reviewed.

2. MODEL AND FORMULATION

A further interesting nanodevice is a ring of many chains, threaded by a magnetic field $B$ as shown in Fig. 1.

![Diagram of a multichain system](image)

Fig. 1 – Schematic view of a multichain system containing $s$ sites.

The transport properties for non-interacting electrons in this system can be described with the help of the tight binding Hamiltonian [7]

$$H = H_{\text{leads}} + H_{\text{chains}} + H_{\text{int}}.$$  (1)
The individual terms in (1) are responsible successively for the leads, the chains and for the tunneling interaction. The corresponding Hamiltonians with nearest neighbor hopping are given by

\begin{equation}
H_{\text{leads}} = \sum_{j=-\infty}^{0} (c_{j}^\dagger c_{j-1} + c_{j-1}^\dagger c_{j}) + \sum_{j=s}^{\infty} (c_{j}^\dagger c_{j+1} + c_{j+1}^\dagger c_{j}),
\end{equation}

\begin{equation}
H_{\text{chains}} = V_{\alpha} \sum_{\alpha=1}^{N_{\alpha}} \left( \sum_{i=1}^{N_{\alpha}} c_{\alpha,i}^\dagger c_{\alpha,i} + \sum_{i=1}^{N_{\alpha-1}} (c_{\alpha,i}^\dagger c_{\alpha,i+1} + c_{\alpha,i+1}^\dagger c_{\alpha,i}) \right),
\end{equation}

\begin{equation}
H_{\text{int}} = \sum_{\alpha=1}^{N} \left( c_{0}^\alpha c_{\alpha,1} + c_{\alpha,1}^\dagger c_{0} + e^{i\theta_{\alpha}} c_{\alpha,N_{\alpha}}^\dagger c_{\alpha} + e^{-i\theta_{\alpha}} c_{\alpha}^\dagger c_{\alpha,N_{\alpha}} \right),
\end{equation}

respectively where \( c_{\alpha,i}(c_{\alpha,i}^\dagger) \) are the annihilation (creation) operator of an electron on the site \( i \) of chain \( \alpha \), while \( c_{j}(c_{j}^\dagger) \) stands for the annihilation (creation) operator characterizing the leads. The number of sites in the \( \alpha \)-th chain is \( N_{\alpha} \) \( (\alpha = 1, 2, \ldots N) \), \( N \) being the total number of chains. These chains are attached to two leads at nodes located at \( x = 0 \) and \( x = s > 0 \). Accordingly, we have to account for the magnetic flux \( \Phi_{\alpha} = BS_{\alpha} \), where \( S_{\alpha} \) is the surface enclosed by chains \( \alpha \) and \( \alpha - 1 \) \( (\alpha = 2, 3, \ldots N) \). We shall also assume, for convenience, that the applied voltage is \( V_{\alpha} \).

The present gauge concerns the site \( i = N_{\alpha} \), such that

\begin{equation}
2\pi \frac{\Phi_{\alpha}}{\Phi_{0}} = \phi_{\alpha} - \phi_{\alpha-1},
\end{equation}

for \( \alpha \geq 2 \), where \( \phi_{1} = 0 \). Note that the factors \( \exp(\pm i\phi_{\alpha}) \) should be viewed as a manifestation of the Aharonov–Bohm effect. We can write the energy eigenvalue equations as \( H|\Psi\rangle = E|\Psi\rangle \), by accounting for expansions over orthogonalized Wannier states like

\begin{equation}
|\Psi\rangle = \sum_{j \leq 0, j \geq s} A_{j} |j\rangle + \sum_{\alpha=1}^{N} \sum_{i=1}^{N_{\alpha}} A_{\alpha,i} |i, \alpha\rangle,
\end{equation}

where \( |j\rangle = c_{j}^\dagger |0\rangle \) and \( |i, \alpha\rangle = c_{\alpha,i}^\dagger |0\rangle \). Of course, one has \( \langle j | j' \rangle = \delta_{jj'} \) and \( \langle i', \alpha'| i, \alpha \rangle = \delta_{ii'}, \delta_{\alpha\alpha'} \).

This leads to discrete equations like

\begin{equation}
EA_{j} = A_{j+1} + A_{j-1},
\end{equation}
for \( j \leq -1 \) and \( j \geq s + 1 \), as well as to

\[
EA_0 = \sum_{\alpha} A_{\alpha,0} + A_{-1},
\]  
\[
EA_{s} = \sum_{\alpha} e^{-i\theta_{\alpha}} A_{\alpha,N_{\alpha}} + A_{s+1},
\]  
\[
(E - V_{\alpha}) A_{\alpha,0} = A_{0} + A_{\alpha,1},
\]  
\[
(E - V_{\alpha}) A_{\alpha,N_{\alpha}} = A_{\alpha,N_{\alpha} - 1} + e^{i\theta_{\alpha}} A_{s},
\]  

and

\[
(E - V_{\alpha}) A_{\alpha,i} = A_{\alpha,i+1} + A_{\alpha,i-1},
\]

where \( 2 \leq i \leq N_{\alpha} - 1 \). Extrapolating to \( i = 1 \) and \( i = N_{\alpha} \) yields

\[
A_{0} = A_{\alpha,0},
\]

and

\[
A_{s} = e^{-i\theta_{\alpha}} A_{\alpha,N_{\alpha} + 1},
\]

Next, we have to resort to plane waves like

\[
A_{j} = A \cdot e^{-ik(j-s)} + R \cdot e^{ik(j-s)},
\]

and

\[
A_{j} = e^{-ikj}, \quad \text{for } j \geq s \text{ and } j \leq 0,
\]

where \( k = \cos^{-1}(E/2) \) is the wave vector of a wave function with energy \( E \), \( A \) is the amplitude of the incident wave, while \( R \) stand for the amplitude of the reflected wave. The transmission coefficient which measures the transparency of the system is defined as \( |\mu|^{2} = 1/|A|^{2} \).

Next, the wave function coefficients in the \( \alpha \)-th chain can be expressed as a linear combination of transmitted and reflected plane waves via

\[
A_{\alpha,l} = A_{\alpha} e^{il\theta_{\alpha}} + R_{\alpha} e^{-il\theta_{\alpha}}, \quad \text{for } 0 \leq l \leq N_{\alpha}.
\]

The coefficient at the left lead node \( j = 0 \),

\[
A_{\alpha} + R_{\alpha} = 1.
\]

in accord with (15)–(17). It is also clear that \( k_{\alpha} = k \) when the applied voltage is zero. Similarly, we can calculate the coefficient at the right node \( j = s \), via

\[
A_{\alpha} e^{i\theta_{\alpha}}(N_{\alpha} + 1) + R_{\alpha} e^{-i\theta_{\alpha}}(N_{\alpha} + 1) = (A + R)e^{i\theta_{\alpha}},
\]
which produces the solutions for all chains $\alpha$ as

$$A_\alpha = \frac{e^{-ik_\alpha (N_\alpha + 1)} - (A + R)e^{ik_\alpha}}{2i\sin(k_\alpha (N_\alpha + 1))}, \quad (20)$$

and

$$R_\alpha = \frac{e^{ik_\alpha (N_\alpha + 1)} - (A + R)e^{ik_\alpha}}{2i\sin(k_\alpha (N_\alpha + 1))}, \quad (21)$$

in terms of $A$ and $R$.

From the Schrödinger difference equations at the lead nodes 0 and $s$ we obtain

$$E = e^{ik} + \sum_{\alpha=1}^{N} \left( A_\alpha e^{ik_\alpha} + R_\alpha e^{-ik_\alpha} \right), \quad (22)$$

$$E(A + R) = A \cdot e^{-ik} + R \cdot e^{ik} + \sum_{\alpha=1}^{N} e^{-ik_\alpha} \left( A_\alpha e^{ik_\alpha N_\alpha} + R_\alpha e^{-ik_\alpha N_\alpha} \right). \quad (23)$$

Inserting (20) and (21) into equations (22) and (23) we obtain

$$E = c_0 + (A + R) f_0 + e^{ik}, \quad (24)$$

and

$$E = (A + R) c_0 + f_0 + A \cdot e^{-ik} + R \cdot e^{ik}, \quad (25)$$

where

$$c_0 = \sum_{\alpha} \frac{\sin(k_\alpha N_\alpha)}{\sin(k_\alpha (N_\alpha + 1))}, \quad (26)$$

and

$$f_0 = \sum_{\alpha} \frac{\sin k_\alpha e^{ik_\alpha}}{\sin k_\alpha (N_\alpha + 1)}. \quad (27)$$

Now we can establish the $A$ parameter as

$$A(k) = A(k + 2\pi) = \frac{|f_0|^2 - (c_0 - e^{-ik})^2}{2f_0 \cos k}. \quad (28)$$

Accordingly, the transmission coefficient can be calculated as

$$|t|^2 = \frac{1}{|A|^2} = \frac{4 \cos^2 k |f_0|^2}{||f_0|^2 - (c_0 - e^{-ik})^2||^2} = \frac{4 \cos^2 k |f_0|^2}{1 + 4|c_0|^2 + 1}, \quad (29)$$

where
\( \Gamma = \Gamma (k, \beta, N_\alpha) = \left( |f_0|^2 - c_0^2 \right)^2 + 2 \left( |f_0|^2 - c_0^2 \right) \left( 2c_0 \cos k - 2 \cos^2 k + 1 \right). \)  \hspace{1cm} (30)

Equation (29) stands for the general analytical expression for the transmission coefficient in multichain nanorings. Next we have to realize that the solutions of the equations \( \Gamma (k, \beta, N_\alpha) = 0 \) serve as candidates to the derivation of maximum values of \( |t|^2 \). This amounts to solve the equations \( c_0 = \pm |f_0| \). One would then obtain the limit

\[ |t|^2 \to 4 \cos^2 k - \frac{c_0^2}{1 + 4c_0^2}. \]  \hspace{1cm} (31)

By virtue of \( \Gamma \to 0 \), which leads to the upper bound

\[ |t|^2 \leq \cos^2 k \leq 1, \]  \hspace{1cm} (32)

proceeding irrespective of \( k \), as one might expect. Further simplifications can be done if the chain lengths are equal to \( N_1 = N_2 = \ldots = N_N = L \), for which the applied voltage is zero. Then we can insert \( \phi_\alpha = 2\pi(\alpha - 1)\beta \) and \( \Phi_\alpha = \Phi \equiv \beta \Phi_0 \) for \( \alpha \geq 2 \). So, \( f_0 \) and \( c_0 \) become

\[ f_0 = f_0(\beta) = \frac{\sin k \sin (\pi N\beta)}{\sin k (L + 1) \sin (\pi \beta)} e^{i\pi(N-1)\beta}, \]  \hspace{1cm} (33)

\[ c_0 = \frac{N \sin kL}{\sin k (L + 1)}, \]  \hspace{1cm} (34)

respectively, where \( k_\alpha = k \) and \( f_0(\beta + 2) = f_0(\beta) \). The periodicity characterizing irrespective of \( N \) the magnetic flux dependence of \( f_0 \) is given by a double flux quantum \( \Phi_0 \). The electronic conductance can be also be directly computed from the transmission coefficient via Landauer formula [13]

\[ G(E) = \frac{|t(E)|^2}{1 - |t(E)|^2}, \]  \hspace{1cm} (35)

at a Fermi energy \( E_F = E \).

\[3. \hspace{1cm} \text{RESULTS} \]

In the presence of a magnetic flux we obtain magneto-oscillations, which are much more complicated than these known in the usual Aharonov–Bohm
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rings determined by the phase shifts in the various propagation paths [7]. In Fig. 2 we show the characteristics of the transmission coefficient on the applied magnetic field. Sharp spike occur when both $N_\alpha$ and the magnetic flux exhibit large values, as indicated in Fig. 2b. We find a remarkable change of the transmission in the case when the areas between neighboring chains enclose equal magnetic flux. The transmission becomes more and more sparsely due to the increase of the magnetic flux through the system.

In Fig. 3 we show the electronic conductance versus the magnetic flux for equal chains with the same nearest neighbor path areas. In this case the curves

![Fig. 2](image1.png)

**Fig. 2** – The characteristics of the transmission coefficient obtained in the presence of a magnetic field. The structure consists of $N = 4$ channels of lengths a) $N_\alpha = 10$; b) $N_\alpha = 100$, $\alpha = 1, 2, 3, 4$ and the magnetic flux threading the system is: a) 0.1; b) 2.0.

![Fig. 3](image2.png)

**Fig. 3** – The electronic conductance versus magnetic flux for a multichain system with equal chain lengths $N_\alpha = 200$, $\alpha = 1, 2, 3, 4$ and electron energy $E = 1.1$. The chain numbers are a) $N = 3$, b) $N = 10$. 
show periodic quantum-magnetic oscillations governed by the field dependence which enters \( f_0 \) via Eqs. (33)–(34), finally leading to \((N-1)\Phi_0\). One can easily deduce the relation between the oscillation period and the distribution of the magnetic fluxes by noticing that the phase shift for every chain must be an integer times \(2\pi\). We may conclude that even a small variation of the chain lengths causes abrupt changes in the conductance oscillation patterns.

The application of a magnetic field, can induce characteristic changes in the phase coherence of the electronic wavefunctions [14] which in turn induces various magneto-oscillation periodicities and interference phenomena by varying the distribution of the relative magnetic flux through the structure. We also found abrupt changes in the plot of the conductance versus the magnetic-flux if the length distribution of the system is modulated, which is useful to distinguish even slight chain length variations. The question of whether ring structures can be extracted from oscillation patterns remains open for further discussions. So far we have just to recall that the oscillations of total persistent currents in Aharonov-Bohm rings exhibit a period doubling when passing from an odd number of electrons to an even one [6].

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