NEW PERTURBATION EXPANSIONS IN QUANTUM CHROMODYNAMICS AND THE DETERMINATION OF $\alpha_s$

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We consider a new class of perturbation expansions, which incorporate in a systematic way the available information about the divergent character of the perturbation series in QCD. The new expansion functions, which replace the powers of the coupling, are defined by the analytic continuation in the Borel plane, achieved through an optimal conformal mapping. We consider several possibilities of implementing the known behaviour of the Borel transform near the leading singularities in the complex plane. The corresponding expansions have the same asymptotic convergence rate, but differ at low orders. We show that the new expansions allow a precise determination of the strong coupling $\alpha_s$ from the hadronic decays of the $\tau$ lepton.

1. INTRODUCTION

As it is known, in 1952 Dyson obtained the famous result [1] that the renormalized perturbation series in Quantum Electrodynamics is divergent. During the subsequent decades, similar results have been obtained [2]-[9] for most of the physically interesting field theories, including Quantum Chromodynamics, the modern theory of strong interactions. Dyson’s proposal to regard a divergent series as asymptotic to the function sought for is nowadays generally adopted. This set a challenge for a radical reformulation of perturbation theory. The Feynman diagrams and renormalization theory yield, at least in principle, the values of all the expansion coefficients. They can tell us whether the series is convergent or not, but what we want to know is under what conditions the physical function can be determined from them. If the series were convergent, the knowledge of all the expansion coefficients would uniquely determine the expanded function. On the other hand, there are infinitely many functions having the same asymptotic expansion. A crucial task is to find effective additional inputs that would be able to
reduce or, if possible, remove the ambiguity. For instance, general properties such as causality and unitarity may be useful. For a recent discussion of the ambiguity of field correlators represented by asymptotic perturbation expansions see [10].

Along these lines, a possibility of exploiting better the low order expansion coefficients known at present from Feynman diagram is to combine them with the available information on the large-order behaviour of the expansion. Such an attempt was made in [11], where a new expansion was defined by the analytic continuation in the Borel complex plane achieved by a conformal mapping\(^1\). Specifically, we used an optimal conformal mapping, as defined in [12]. As shown in [11, 13, 14], the new perturbation expansion separates the divergent character of the series from a part of the intrinsic ambiguity of the perturbation theory, which is generated by the infrared regions of the Feynman diagrams. This ambiguity is solved by choosing a prescription included in the definition of the expansion functions.

In the present paper we discuss these ideas using for illustration the so-called Adler function [15]. This function is of interest in QCD for the determination of the strong coupling \(\alpha_s\) at a relatively low scale, the mass of the \(\tau\) lepton [16]-[22]. In section 2 we briefly review the standard perturbation expansion of the Adler function in massless QCD. In section 3 we present the new perturbation expansion proposed in [11] and in section 4 we investigate a more general class of expansions functions, which have the same asymptotic convergence rate but differ at low orders. The convergence of the new expansions is checked using a realistic model proposed recently in the literature [20]. In section 5 we briefly discuss the determination of \(\alpha_s\left(m_{\tau}^2\right)\), confirming the precise value reported recently in [21].

2. STANDARD PERTURBATION EXPANSION

The Adler function [15] in massless QCD is the derivative

\[
D(s) = -s \frac{d\Pi(s)}{ds},
\]

where \(\Pi(s)\) is the invariant amplitude of the correlator

\[
i \left< \int d^4xe^{iqx} \right| \mathcal{T}\left\{V_{\mu}(x)\ V_{\nu}^\dagger(0)\right\} \left| 0 \right>= (g_{\mu\nu}q^2 - g_{\mu\nu})\Pi(s).
\]

Here \(s = q^2\) is the energy squared and \(V_{\mu}\) is the vector current for light quarks.

\(^1\) For QCD, the idea of a conformal mapping in the Borel plane was suggested in [7] and was applied in several works (for references see [11]).
In perturbative QCD, the reduced Adler function \( \hat{D}(s) = D(s) - 1 \) is written formally as the renormalization-group improved series

\[
\hat{D}(s) = \sum_{n=1}^{\infty} K_n \left( a_s(s) \right)^n ,
\]

where \( a_s(s) \equiv \frac{\alpha_s(s)}{\pi} \) is the running coupling. Explicit calculations of Feynman diagrams in \( \overline{\text{MS}} \) renormalization scheme with three flavours \( n_f = 3 \), predict the first four coefficients [23]-[25]

\[
K_1 = 1, \quad K_2 = 1.639, \quad K_3 = 6.371, \quad K_4 = 49.076,
\]

while an estimate of the next coefficient

\[
K_5 \approx 283
\]

is available [20].

As discussed above, the series (3) is divergent, the coefficients displaying a factorial asymptotic increase, \( K_n \sim n! \) [8,9]. In writing (3) we follow the convention often adopted in physical papers, writing the sign of equality even if the series on the right hand side is divergent and the equality is impossible. Analogous series in the present paper are understood in the same sense.

The running coupling \( a_s(s) \) is obtained by setting the scale \( \mu^2 = s \) in the renormalized coupling \( a_s(\mu^2) = a_s(\mu^2)/\pi \), which satisfies the renormalization group equation

\[
\mu^2 \frac{da_s(\mu^2)}{d\mu^2} = \beta \left( a_s(\mu^2) \right) = -\sum_j \beta_j \left( a_s(\mu^2) \right)^{j+2} .
\]

The first two coefficients of the \( \beta \) function are universal (scheme-independent), and the next two were calculated in the \( \overline{\text{MS}} \) scheme with \( n_f = 3 \) to four loops [26, 27]. The known coefficients are:

\[
\beta_0 = 9/4, \quad \beta_1 = 4, \quad \beta_2 = 10.06, \quad \beta_3 = 47.23.
\]

### 3. NEW PERTURBATION EXPANSIONS

To define a new series, we start from the Borel transform \( B(u) \) of the Adler function, defined by the power series

\[
B(u) = \sum_{n=0}^{\infty} b_n u^n ,
\]

with \( b_n \) defined in terms of the perturbative coefficients \( K_n \) by:
According to present knowledge, the function $B(u)$ has branch point singularities in the $u$-plane, along the negative axis – the ultraviolet (UV) renormalons – and along the positive axis – the infrared (IR) renormalons [8,9]. Specifically, the branch cuts are situated along the rays $u \leq -1$ and $u \geq 2$. The nature of the first branch points was established in [6] and in [28] (see also [20]). Thus, near the first branch points, i.e. for $u \sim -1$ and $u \sim 2$, respectively, $B(u)$ behaves as

$$B(u) \sim \frac{r_1}{(1+u)\gamma_1}, \quad B(u) \sim \frac{r_2}{(1-u/2)\gamma_2},$$

where the residues $r_1$ and $r_2$ are not known, but the exponents $\gamma_1$ and $\gamma_2$ can be calculated using renormalization group invariance [6,28,20], and have the values

$$\gamma_1 = 1.21, \quad \gamma_2 = 2.58.$$  \hspace{1cm} (11)

The series (3) can be formally written as the Borel-Laplace transform

$$\hat{D}(s) = \frac{1}{\beta_0^2} \int_0^\infty e^{-u/\beta_0(s)} B(u) \, du.$$ \hspace{1cm} (12)

Actually, due to the singularities of $B(u)$ along the positive axis, the integral (12) does not exist. The ambiguity in the choice of prescription is often used as a measure of the uncertainty of the calculations in perturbative QCD. It is convenient to define the integral by the Principal Value (PV) prescription:

$$\hat{D}(s) = \frac{1}{\beta_0} \text{PV} \int_0^\infty e^{-u/\beta_0(s)} B(u) \, du,$$ \hspace{1cm} (13)

where

$$\text{PV} \int_0^\infty f(u) \, du = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_0^\infty \left[ f(u + \varepsilon) \, du + f(u - \varepsilon) \, du \right].$$

As discussed in [29] the PV prescription is the best choice if one wants to preserve as much as possible the analyticity properties of the correlators in the $s$-plane, which are connected with causality and unitarity.

In order to define a new perturbative expansion of the Adler function we shall apply the method of conformal mappings [12]. This method is not applicable to the series (3), because $\hat{D}(s)$ (regarded as a function of $a$, $s$) is singular at the point of
expansion \(a_n(s) = 0\). The method can, on the other hand, be applied to (8), because \(B(u)\) is holomorphic at \(u = 0\).

We note that the expansion (8) converges only in the disk \(|u| < 1\). A series with a larger domain of convergence can be obtained by expanding \(B(u)\) in powers of a new variable. As demonstrated in [12], the optimal variable, which leads to the best asymptotic convergence rate, coincides with the function that performs the conformal mapping of the whole analyticity domain of the expanded function onto a disk in the new complex plane.

It is generally assumed that \(B(u)\) is analytic in the \(u\)-plane cut along the real axis for \(u \geq 2\) and \(u \leq -1\) (for more remarks see [10, 21]). Then the optimal variable defined in [12] reads [11]

\[
\frac{1}{2} \sqrt{1 + u - \frac{1 - u}{2}} \quad \text{or} \quad w(u) = \frac{1}{2} \sqrt{1 + u + \frac{1 - u}{2}}. \tag{14}
\]

This function maps the \(u\)-plane cut for \(u \geq 2\) and \(u \leq -1\) onto the unit disk \(|w| < 1\) in the complex plane \(w = w(u)\), such that \(w(0) = 0\), \(w(2) = 1\) and \(w(-1) = -1\). Then, the expansion

\[
B(u) = \sum_{n=0}^{\infty} d_n w^n \tag{15}
\]

converges in the whole disk \(|w| < 1\) [12, 11]. Moreover, as shown in [12], the expansion (15) has the best asymptotic rate of convergence compared to all the expansions of the function \(B(u)\) in powers of other variables.

The series (15) can be used to define an alternative expansion of \(\hat{D}(s)\). This is obtained formally by inserting (15) into (13) and interchanging the order of summation and integration. Thus, we adopt the modified expansion, defined as [11]-[14]:

\[
\hat{D}(s) = \sum_{n=0}^{\infty} d_n W_n(s), \tag{16}
\]

in terms of the expansion functions

\[
W_n(s) = \frac{1}{\beta_0} \text{PV} \int_0^{\beta} e^{-w(\beta, s)} w^n \, dw. \tag{17}
\]

We emphasize that the expansion (15) exploits only the location of the leading singularities in the Borel plane. The series is expected to describe also the nature of the singularities if a large number of terms is used. However, since the behaviour near the first singularities is known, cf. eq. (10), it is convenient to incorporate it explicitly. This is achieved, for instance, by expanding the product \((1 + w)^{2\gamma_1} (1 - w)^{2\gamma_2} B(u)\) in powers of the variable \(w\):
Here we took into account the fact that from the expression (14) of \( w = w(u) \) it follows that \( (1 - u/2) \sim (1 + w)^2 \) and \( (1 + u) \sim (1 - w)^2 \) near the points \( w = -1 \) and \( w = 1 \), respectively.

The expansion (18) converges in the whole disk \(|w| < 1\), i.e. in the whole cut complex \( u\)-plane. Moreover, since the singular behaviour of \( B(u) \) at the first branch points is compensated by the first factors in (18), the series is expected to converge faster than (15). Also, the behaviour near the first singularities holds even for truncated expansions, which are used in practice. This suggests the definition of the new expansion

\[
\hat{D}(s) = \sum_{n=0}^\infty c_n W_n(s),
\]

where the expansion functions are

\[
W_n(s) = \frac{1}{\beta_0} \text{PV} \int_0^\infty \frac{w^n}{(1+w)^{2\gamma_1} (1-w)^{2\gamma_2}} \, du,
\]

with \( w = w(u) \) defined in (14). Actually, as discussed in [11, 21], the inclusion of the explicit behaviour near the leading singularities is not unique. The problem will be discussed briefly in the next section.

The expansions (16) and (19) reproduce the coefficients \( K_n \) of the usual expansion (3), when the functions (17) and (20) are expanded in powers of the coupling. As shown in [14], the new expansion functions are formally represented by divergent series in powers of the coupling, much like the expanded correlator itself.

A detailed comparison of the standard and the new expansions (16) and (19) was performed in [21], using as reference a realistic model for the Adler function proposed in [20]. This model, which expresses the Borel transform in terms of a few singularities, allows the calculation of the exact Adler function and of its perturbative approximants. To illustrate the comparison, we shall use the contour integral

\[
\delta^{(0)} = \frac{1}{2\pi i} \oint \frac{ds}{s} \omega(s) \hat{D}(s)
\]

where \( s_0 = m_t^2 \) and \( \omega(s) = 1 - 2s/s_0 + 2(s/s_0)^3 - (s/s_0)^4 \) is a kinematical factor. The quantity \( \delta^{(0)} \) can be determined experimentally in terms of the total hadronic width of the \( \tau \) lepton [16]. Therefore, the integral (21) is of interest for the experimental determination of the strong coupling from the hadronic \( \tau \)-decays.
To perform the comparison, we calculate $\delta^{(0)}$ using the exact expression of the Adler function in the model [20], and its expansions truncated at a finite order $N$. The running coupling $a_s(s)$ is obtained by integrating numerically the renormalization group equation (6) step by step along the circle $s = m^2$, in terms of a reference value, taken to be $a_s(m^2)$. This procedure, which avoids the large imaginary logarithms appearing in expansions at a fixed scale is known as “contour improved” expansion of $\delta^{(0)}$ [16]. For illustration we use $a_s(m^2) = 0.34/\pi$, as in [20, 21].

Fig. 1 – Values of $\delta^{(0)}$ for the model discussed in [20], obtained with the standard and the new expansions, as a function of the perturbative order $N$. The horizontal band is the exact value from [20] with an experimental error. The calculations are performed with $a_s(m^2) = 0.34/\pi$.

The results are shown in Fig. 1, where the gray band represents the exact result, to which we attached for convenience an uncertainty equal to the experimental error (see section 5). As seen in Fig. 1, the standard expansion is not able to reproduce the exact value at low truncation orders $N$. Moreover, for higher $N$ the results start to oscillate wildly, due to the divergent character of the series. On the other hand, our new expansions (16) and (19) reproduce more closely the exact value. The expansion (19), which includes explicitly the singular behaviour at the first singularities, gives very good values even for low $N$, while the expansion (16) gives a good approximation only at larger truncation orders $N$. This is due to the fact that, in the model proposed in [20], the strength of the first singularities, expressed by the residues in (10), is quite important. Therefore, a large number of terms are required in order to describe them, if the singularity is not explicitly factorized. In other models, with milder leading singularities, the expansion (16) approximates well the exact results even at low values of $N$. Both expansions, (19) and (16), are actually much better than the standard expansion (3) in powers of the coupling.
4. OTHER ADMISSIBLE EXPANSIONS

As remarked above (see also [11, 21]), while the expansion (15) is unique, the explicit inclusion of the first singularities of $B(u)$ contains some arbitrariness. The description of the singularities by multiplicative factors is a possibility, but is not a priori necessary. Moreover, the factors are not unique. For a large number of terms in the expansion the form of these factors is irrelevant, but at low orders one prescription may be better than another. In (19) the dominant behaviour was included by simple singular factors expressed in the $w$ variable. In this section we briefly discuss several other possible expansions. A more complete analysis will be presented elsewhere.

To construct alternative approximants, we first remark that the dominant behaviour may be included by singular factors expressed in the $u$ variable, instead of $w$. One may further multiply the leading factors by other functions analytic in the $u$-complex plane cut along the real axis for $u \geq 2$ and $u \leq -1$. In particular, one may consider singularities on an unphysical Riemann sheet, or placed at $u = 3$ and $u = -2$, which are expected to occur as next-to-leading renormalons in the physical case. The additional factors can be expressed either in the variable $u$ or in the variable $w$. We consider for illustration a set of new expansions written in the compact form

$$\hat{D}(s) = \sum_{n=0}^{\infty} f_{n,k} W_n^k(s),$$

where the expansion functions are defined as

$$W_n^k(s) = \frac{1}{\beta_0} PV \int_0^\infty e^{-u/(\beta_0 \mu(s))} \frac{w^n}{F_k(u)} du,$$

with the specific choices

$$F_1(u) = (1-u)^{7/2} (1-u/2)^{7/2},$$

$$F_2(u) = (1+w)^{2\gamma_1} (1-w)^{2\gamma_1} (2-w)^2 (3-w)^2,$$

$$F_3(u) = (1+w)^{2\gamma_1} (1-w)^{2\gamma_2} (1+u/2)^2 (1-u/3)^2,$$

$$F_4(u) = (1+w)^{2\gamma_1} (1-w)^{2\gamma_2} \prod_{j=1}^4 \left(1 - \frac{w}{w_k}\right)^2,$$

$$F_5(u) = (1+w)^{2\gamma_1} (1-w)^{2\gamma_2} \prod_{j=1}^4 \left(1 - \frac{w}{w_k}\right)^{0.5}.$$

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Here \( w \equiv w(u) \), where the function \( w(u) \) is defined in (14), and we denoted by 
\( w_1 = w(3 + i \varepsilon), \quad w_2 = w(3 - i \varepsilon), \quad w_3 = w(-2 + i \varepsilon), \quad w_4 = w(-2 - i \varepsilon) \) the positions the points \( u = 3 \pm i \varepsilon \) and \( u = -2 \pm i \varepsilon \) on the circle \(|w|=1\) in the \( w \)-plane. In the absence of further information, the exponents of the additional singularities in \( F_2(u), F_3(u), F_4(u) \) and \( F_5(u) \) are arbitrary.

In Fig. 2 we test the efficiency of the above approximants, using the model \([20]\) and \( a_s(m_T^2)=0.34/\pi \) as in the previous section. The results shown in the left panel confirm that at large \( N \) all the choices become equivalent, as expected\(^2\). At low \( N \) the differences between various approximations are more pronounced, but starting from \( N = 5 \), which corresponds to the present status in QCD, the range becomes rather narrow. In particular, for \( N = 5 \) the expansion (19) adopted in [21] gives the central value, the spread around it being \( \pm 0.006 \), close to the experimental uncertainty (we recall that for comparison we attached an “experimental” error to the exact result, as shown by the grey band in Fig. 2). We shall discuss in the next section the implications of these results on the determination of \( \alpha_s \).

5. DETERMINATION OF \( \alpha_s \)

The precise determination of \( \alpha_s \) from the hadronic decays of the \( \tau \) lepton is one of the most important results in perturbative QCD (for a recent review see [22]). The problem was revisited recently [19, 20, 21], after the calculation of \( \alpha_s \) to higher orders.

\(^2\) A possible divergency at still higher values of \( N \), not excluded in principle, is discussed in [13, 14, 21].
the Adler function to four loops [25], the same order at which the $\beta$ function of the
renormalization group equation is known [26, 27].

As mentioned in the previous section, the quantity relevant for the extraction of $\alpha_s\left(m^2\right)$ is the integral (21). We shall adopt the phenomenological value quoted in [20]:

$$\delta^{(0)}_{\text{phen}} = 0.2042 \pm 0.0050. \quad (29)$$

The determination of $\alpha_s\left(m^2\right)$ then amounts to solving the equation $\delta^{(0)} = \delta^{(0)}_{\text{phen}}$, with $\delta^{(0)}$ calculated from (21) using various theoretical expansions of $\hat{D}(s)$. We use as input the known coefficients $K_\alpha$ from (4) and $K_5$ from (5), and the running coupling expressed in terms of $\alpha_s\left(m^2\right)$ by solving numerically the renormalization group equation (6) along the integration contour.

Taking the average over the various functional forms discussed in the previous sections, we obtain

$$\alpha_s\left(m^2\right) = 0.320 \pm 0.012, \quad (30)$$

where the error includes the experimental uncertainty quoted in (29), the effect of a 50% variation of the coefficient $K_5$, the uncertainty of the renormalization scale [21], and an additional error of $\pm 0.003$ due to the freedom in choosing the expansion.

6. CONCLUSIONS

The new perturbation expansion for QCD observables, proposed in [11], is based on an optimal expansion of the Borel transform, which reproduces order by order the known perturbative coefficients calculated from Feynman diagrams and converges in the whole complex $u$-plane cut along two lines of the real axis. While the optimal expansion variable $w$ is unique, a freedom exists in the way of including additional information, like the known behaviour of the Borel transform near the first branch points. The inclusion of this behaviour is neither unique nor precisely formulated and we have no optimal method, in contradistinction to the choice of the optimal $w$, which is based on an optimality theorem [12].

In the present paper we investigated a set of possible ways of implementing the behaviour of the Borel transform at the leading singularities and tested their properties in the frame of a model proposed recently in [20]. The results show that, for the number of perturbative terms calculated at present from Feynman diagrams in QCD, the various approximants give consistent results.

Our analysis shows that the freedom in the implementation of the known singular behaviour of the Borel transform does not affect practically the extraction of $\alpha_s$ from $\tau$ hadronic decays: the central value of $\alpha_s\left(m^2\right)$ is unmodified, the only
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...effect being an additional error on of about $\pm 0.003$, comparable with the experimental error. Our final result is given in (30). A detailed analysis, including a more general class of approximants and the investigation of alternative physical models for the QCD correlator, will be presented in a future work.

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