This is a brief review of our recent work related to the thermal and elastic properties of amorphous materials and disordered crystals. The work is based on a model for the interaction between two-level systems and elastic displacement fields in solids. We apply the model to the calculation of the thermal properties of nanoscopic amorphous membranes and sound attenuation in disordered crystals. We explain an old puzzle related to the anisotropy of the glassy properties of disordered crystals.

**Key words:** Disordered solids; Anelasticity, internal friction, stress relaxation, and mechanical resonances; Phonon-defect interactions.

**PACS:** 61.43.-j; 62.40.+i; 63.20.kp

### 1. INTRODUCTION

It is an honor and a pleasure for me to contribute to this volume, dedicated to the 80th anniversary of Prof. Oliviu Gherman.

We shall review here briefly the main results we have obtained in the past few years in the description of the low temperature properties of disordered solids. These properties are determined by the dynamical defects in the solid, which are modeled by an ensemble of two-level systems (TLS). We proposed a new model of two-level systems which allowed us to describe in a very general way the interaction of these systems with the elastic deformation of the solids and therefore with the phonons and the elastic waves [1, 2]. Moreover, we also explained within this model the anisotropic glassy properties of disordered crystals [3, 4], a phenomenon observed decades ago, but which was eventually never explained [5–9].

The work was initially motivated by the development of ultra-sensitive detectors of electromagnetic radiation. These detectors consist roughly of an ultra-thin metallic film, called the *thermal sensing element* (TSE), which is deposited on a free-standing dielectric membrane. Typically, the membranes are 100-200 nm thick, made of amorphous, low stress silicon nitride ($\text{SiN}_x$), grown on a crystalline (100)-oriented Si wafers [10, 11]. These detectors function in a temperature range below 1 K and at these temperatures the thermodynamic and heat transport properties of the...
membranes are specifically important since the heat release from the TSE becomes the bottleneck for the device performance [12–14].

At these temperatures the thermal and transport properties of the membranes are rather unusual. For example in many experiments the heat conductivity, \( \kappa \), of wide, thin membranes or long, narrow and thin bridges is proportional to \( T^p \), where \( p \) takes values between 1.5 and 2 [11, 15–17].

The specific heat of mesoscopic membranes is more difficult to measure. It can, however, be extracted from the amplitude of the temperature oscillations in AC measurements and it appears also to be proportional to \( T^p \), where \( p \) lies roughly between 1 and 2 [11].

To explain this behavior, we first observe that below 1 K the dominant phonon wave-length is of the order of 100 nm and therefore is comparable to the supporting membrane thickness. This implies that as the temperature decreases, a crossover from a three-dimensional to a two-dimensional phonon gas distribution appears in the membrane, which may eventually explain the decrease of the exponent, \( p \), of the temperature dependence of \( \kappa \) and \( c_V \) from 3 to 2 [17–19]. In very narrow bridges, \( p \) of \( \kappa \) may decrease even to 1.5 below the crossover temperature [19].

Nevertheless, in some experiments the same behavior, \( \kappa \propto T^p \) with \( p \) being roughly 2 [10,11,16,20,21], plus an increase of \( f_c \) with the temperature [10,11], was observed also above the 2D–3D crossover temperature. To explain these features we had to take into account the amorphous structure of the membranes’ material and therefore to include the contribution of the TLSs into the model.

Here we run into another difficulty. Amorphous or glassy materials differ significantly from crystals, especially in the low temperature range, where, for 3D bulks, \( \kappa \propto T^2 \) and \( c_V \propto T \) [22, 23]. These temperature dependences were explained independently by Phillips [24] and Anderson, Halperin and Varma [25] by the presence of the dynamic defects modeled, as mentioned above, by an ensemble of TLSs. A TLS can be understood as an atom, or a group of atoms, which can tunnel between two close minima in configuration space, forming a hybridized doublet state. The presence of such minima is a hallmark of the glassy state. If the energy splitting between these minima is \( \lesssim k_B T \), then the TLS can be excited from its ground state onto the upper level. In this way the TLSs contribute to the heat capacity. TLSs can also scatter phonons and in this way decrease their mean free path and, correspondingly, the heat conductance.

The standard theory of glassy materials, the so called the standard tunneling model (STM), was developed mainly in the 1970s and therefore was concerned mainly with isotropic, bulk 3D materials. This involved a scalar interactions between isotropic TLSs and plane-waves which are not suitable for mesoscopic systems, in which the phonon modes are complex superpositions of waves of different polariza-
An effective double-well potential of a TLS is depicted in Fig. 1 [24, 25]. Written in the 2D Hilbert space spanned by the ground states of the two wells, the effective Hamiltonian of this TLS reads
\[ H_{TLS} = \Delta \sigma_z - \Lambda \sigma_x \]
with \( \sigma_x \) and \( \sigma_z \) Pauli matrices and \( \Lambda \) describing the tunneling between the two wells. In general, \( \Delta \) is called the asymmetry of the potential and \( \Lambda \) is called the tunnel splitting. The Hamiltonian may be diagonalized by an orthogonal transformation \( O \),
\[ H'_{TLS} \equiv O^T H_{TLS} O = (\epsilon/2) \sigma_z, \]
where \( \epsilon \equiv \sqrt{\Delta^2 + \Lambda^2} \) is the excitation energy of this TLS and by the superscript \( T \) we denote in general the transpose of a matrix. Since the TLS can be in only two states, we shall denote the ground state by \( |\downarrow\rangle \) and the excited state by \( |\uparrow\rangle \). Phonons at frequencies close to the level splitting \( \epsilon \) are strongly scattered by TLSs.

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The Hamiltonian parameters \( \Delta \) and \( \Lambda \) are distributed with the density \( V P(\Delta, \Lambda) \), where \( V \) is the volume of the solid. According to the standard tunneling model (STM), \( P(\Delta, \Lambda) = P_0 / \Lambda \), where \( P_0 \) is a constant. If expressed through the variables \( \epsilon \) and \( u \equiv \Lambda / \epsilon \), the distribution function is
\[ P(\epsilon, u) = P_0 / \left( u \sqrt{1 - u^2} \right). \]

The strain caused by a phonon or any other kind of distortion of the material adds a perturbation to \( H_{TLS} \), which we denote by \( H_1 \). The total Hamiltonian, \( H = H_{TLS} + H_1 \), enables us to describe the coupling between a TLS and the phonon field. In the STM the variation of the off-diagonal elements of \( H_{TLS} \) is zero, so
\[ H_1 = (\delta/2)\sigma_z \] [23–27]. The perturbation \( \delta \) is linear in the strain field, \( S_{ij} \), [23, 27] and in general may be written as \( \delta \equiv 2\gamma_{ij}S_{ij} \). (Here, as everywhere in this paper, we assume summation over repeated indices.) The 3 \times 3 symmetric strain tensor is defined as \( S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \), with \( u_i \), \( i = 1, 2, 3 \), being the components of the displacement field.

We are interested in the thermal properties of the solid, so we study the effect of a time-dependent Hamiltonian \( H_1 \), produced by the interaction of the TLS with a phonon field. This interaction produces transitions of the TLS between its eigenstates in the unperturbed state.

In the STM the TLS interacts with 3D, transversally or longitudinally polarized, plane waves, which have all the components of the strain tensor proportional to the absolute value of the wave vector, \( k \). The longitudinal wave will produce a compressional strain \( (S_{yz} = S_{zx} = S_{xy} = S_{zy} = S_{xz} = S_{yx} = 0) \) and the transversal wave will produce a shear strain \( (S_{xx} = S_{yy} = S_{zz} = 0) \). In this case, the expression for \( \delta \) is always reduced to \( \delta \equiv 2N_\sigma\gamma_\sigma k_\sigma \), where \( \gamma_\sigma \) is called the deformation potential parameter or coupling constant.

With these ingredients and denoting by \( n_{k,\sigma} \) the population of the phonon mode of wave-vector \( k \) and polarization \( \sigma \) and by \( f_\epsilon \) the population of the excited TLS levels of energy \( \epsilon \), we obtain the phonon emission and absorption rates by the TLS,

\[ \Gamma_{em}(\epsilon) = \gamma_\sigma^2 \frac{\pi k}{\rho c_\sigma} \frac{A^2}{\epsilon^2} (n_{k\sigma} + 1) \delta(hc_\sigma k - \epsilon), \] (1)

and

\[ \Gamma_{abs}(k,\sigma) = \gamma_\sigma^2 \frac{\pi k}{\rho c_\sigma} \frac{A^2}{\epsilon^2} n_{k\sigma} \delta(hc_\sigma k - \epsilon), \] (2)

respectively. In the relaxation time approximation, summing \( \Gamma_{em}(\epsilon) \) over all the phonon modes, we obtain the TLS relaxation time as [23, 26]

\[ \tau_{-1}^{-1} = \left( \frac{\gamma_\sigma^2}{c_t^2} + \frac{2\gamma_\sigma^2}{c_s^2} \right) \frac{A^2 \epsilon}{2\pi \rho h^4} \cdot \coth \left( \frac{\beta \epsilon}{2} \right). \] (3)

Similarly, we obtain the phonon relaxation time by summing over all the TLS states [23, 26]:

\[ \tau_{-1}^{-1} = \frac{\pi \hbar \omega_{k,\sigma}}{h \rho c_\sigma^2} \cdot \gamma_\sigma^2 F_0 \cdot \tanh \left( \frac{\beta \hbar \omega_{k,\sigma}}{2} \right). \] (4)

The low temperature mean free path of a phonon in the amorphous material, \( c_\sigma \tau_{k\sigma} \), can be measured by unsaturated ultrasound attenuation—i.e. the attenuation of ultrasound waves of low enough amplitude, so that only a small fraction of TLSs are
excited out of equilibrium. From here we can extract $\gamma^2 P_0$. Another way to determine $\gamma^2 P_0$ experimentally is by measuring the relative shift in the sound velocity [23,28]:

$$\frac{\Delta c_\sigma}{c_\sigma} = \frac{\gamma^2 P_0}{\rho c_\sigma^2} \ln \left( \frac{T}{T_0} \right), \quad (5)$$

where $T_0$ is a reference temperature at which $\Delta c_\sigma = 0$.

The values obtained for $\gamma^2 P_0$ by these two independent methods can be combined to calculate the heat conductivity

$$\kappa(T) = \frac{\rho k_B}{6\pi} \left( \frac{c_t}{\gamma^2 P_0} + \frac{2c_l}{\gamma^2 P_0} \right), \quad (6)$$

The simplified expression for the perturbation term $\delta$ in the interaction Hamiltonian does not allow us to calculate the interaction of the TLS with an arbitrary strain field. Since, as mentioned before, the elastic modes in mesoscopic systems or in anisotropic solids have rather complicated displacement and strain fields, we need a more general interaction Hamiltonian which should also have all the symmetry requirements of the lattice and of the TLS.

3. THE GENERAL TLS-PHONON INTERACTION HAMILTONIAN

To generalize the TLS Hamiltonian we shall use the full expression for $\delta$:

$$\delta = 2 \gamma_{ij} S_{ij} \equiv 2[\gamma] : [S], \quad (7)$$

where $\gamma_{ij}$ are the components of the $3 \times 3$ tensor $[\gamma]$ and $: \equiv$ is the symbol for dyadic product. Let us now find the general properties of $[\gamma]$.

As follows from Eq. (7), only the symmetric part of the tensor $[\gamma]$ has a physical meaning. Indeed, the dyadic product between a symmetric and an antisymmetric tensor is zero, so the antisymmetric part of $[\gamma]$, even if existent, would not influence the results. We therefore assume $[\gamma]$ to be symmetric. Since all the tensors we have in our model are symmetric, it is more convenient to work using the abbreviated subscript notation, as described for example in Ref. [29]. Let us explain this method briefly.

Let us assume that $[A]$ is a symmetric $3 \times 3$ tensor (i.e. $A_{ij} = A_{ji}$). From its 9 elements, only 6 are independent. To get rid of the redundant 3 elements and also to make the tensors manipulation easier, we can write $[A]$ in the form of a six component vector, $\mathbf{A}$, in the following 2 ways [29]:

$$A_1 \equiv A_{xx}, \quad A_2 \equiv A_{yy}, \quad A_3 \equiv A_{zz}, \quad A_4 \equiv A_{yz}, \quad A_5 \equiv A_{zx}, \quad A_6 \equiv A_{xy} \quad (8)$$
or

\[ A_1 \equiv A_{xx}, \ A_2 \equiv A_{yy}, \ A_3 \equiv A_{zz}, \ A_4 \equiv 2A_{yz}, \ A_5 \equiv 2A_{zx}, \ A_6 \equiv 2A_{xy} \quad (9) \]

Then the dyadic product of two symmetric tensors, \([A] : [B]\), may be written as \([A] : [B] = A^T \cdot B\) if one of the tensors is written in the form (8) and the other one in the form (9). Applying this way of writing to equation (7), we define \(\gamma \equiv (\gamma_{xx}, \gamma_{yy}, \gamma_{zz}, \gamma_{yz}, \gamma_{zx}, \gamma_{xy})^T\) and \(S \equiv (S_{xx}, S_{yy}, S_{zz}, 2S_{yz}, 2S_{zx}, 2S_{xy})\), so that \(\delta = 2\gamma^T \cdot S\).

As in Ref. [29], the subscripts in the abbreviated subscript notations will be denoted by capital letters, \(I, J, K, \ldots\), which run from 1 to 6.

The vector \(\gamma\) should characterize the TLS and its deformability in the presence of a strain field. As explained before, the TLS is imagined as a particle or a group of particles that tunnels from one potential well to another. This tunneling may happen as a translation between the wells, or as a rotation [30]. In either case, there is a direction associated to the TLS, which we call \(\hat{t}\), for example the direction defined by the two potential wells or the axis of rotation. One can expect that the orientation of the TLS (i.e. \(\hat{t}\)) relative to the phonon’s propagation direction and polarization has an effect on the interaction strength. The three components of \(\hat{t}\) are the only co(ntra)variant quantities that describe the TLS, from a very general point of view—i.e. without building a microscopic model of the TLS. With these quantities we can construct the simplest \(3 \times 3\) symmetric tensor

\[
[T] = \begin{pmatrix}
t_x^2 & t_xt_y & t_xt_z \\
t_xt_y & t_y^2 & t_yt_z \\
t_xt_z & t_yt_z & t_z^2
\end{pmatrix} \equiv \hat{t} \cdot \hat{t}^T,
\]

and a general one, \([\gamma] = [R] : [T]\) (i.e. \(\gamma_{kl} = R_{ijkl} T_{ij}\)), with \(R_{ijkl} = R_{ikjl}\) for any \(k\) and \(l\). We shall also choose \(R_{ijkl} = R_{ijlk}\), since the summation \(R_{ijkl} T_{ij}\) allows us to use this simplification. In abbreviated subscript notations, \(T = (t_x^2, t_y^2, t_z^2, 2t_yt_z, 2t_xt_z, 2t_xt_y)^T\), and \(R_{ijkl}\) becomes \(R_{ijl}\) in an obvious way. Then we can write \(\gamma\) as

\[
\gamma \equiv [R]^T \cdot T. \quad (10)
\]

Since \(T\) characterizes the orientation of the TLS, the relevant deformation potential parameters are contained in \([R]\). The tensor \([R]\) is similar to the tensor of elastic stiffness constants from elasticity theory.

Like the elastic stiffness constants, the matrix \([R]\) is determined by the symmetry properties of the lattice. The product \(h_1 \equiv T^T \cdot [R] \cdot S\) is a scalar, so it should be invariant under any rotation of coordinates.
4. EXAMPLES

4.1. ISOTROPIC SOLIDS

A typical amorphous material is isotropic. In this case $[R]$ takes the simplest possible form, $R_{IJ} = 2\xi\delta_{IJ} + \zeta$ for $I, J \leq 3$ and $R_{IJ} = \delta_{IJ}\xi$ for $I$ or $J$ bigger than 3, or, in normal subscripts, $R_{ijkl} = \zeta\delta_{ij}\delta_{kl} + \xi(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The constants $\xi$ and $\zeta$ are analogous to the Lamé constants in elasticity, typically denoted by $\mu$ and $\lambda$. If we plug this tensor into the expressions of the absorption and emission rates, we obtain formulas similar to Eqs. (1) and (2), but with $\gamma_l$ and $\gamma_t$ given by

$$
\gamma_l^2 = \tilde{\gamma}^2 \frac{1}{15} (15 - 40\zeta' + 32(\zeta')^2) \quad \text{and} \quad \gamma_t^2 = \tilde{\gamma}^2 \frac{4}{15}(\zeta')^2,
$$

(11)

where $\tilde{\gamma} \equiv \zeta + 2\xi$, $\mu' = \mu/\tilde{\gamma}$ and $\zeta' = \zeta/\tilde{\gamma}$; this implies $\zeta' + 2\xi = 1$.

From Eqs. (11) we see immediately that $C_l > C_t \geq 0$ for any real $\xi$, as it is observed experimentally in general [1].

This form of $[R]$ have been applied to the calculation of scattering rates of phonons and heat conductivity in amorphous, dielectric membranes in Ref. [2] and to the calculation of the TLS-TLS interaction in Ref. [31].

4.2. CUBIC LATTICE

The tensor $[R]$ for the cubic lattice is very similar to the one for the isotropic material, $[R] = \tilde{\gamma}[r]$, with $[r]$ being the symmetric matrix, $r_{11} = r_{22} = r_{33} = 1$, $r_{12} = r_{13} = r_{23} = r_{21} = r_{31} = r_{32} = \zeta'$, $r_{44} = r_{55} = r_{66} = \xi'$, and the other elements equal to zero. The only difference between the cubic crystal and the isotropic materials being that we do not impose anymore the condition $\zeta' + 2\xi = 1$. Solving the Christoffel equation, we find that a cubic lattice can sustain simple, longitudinally and transversally pozarized plane waves in the crystalographic directions $\langle 100 \rangle$, $\langle 110 \rangle$ and $\langle 111 \rangle$. Along these direction, Topp and coworkers [9] measured the sound attenuation in Ca stabilized zirconium and observed the anisotropy.

Calculating the phonon absorption rates, we obtain in the $\langle 100 \rangle$ direction the longitudinal and transversal coupling constants $\gamma_{l,(100)}^2 = [3 + 4\zeta' + 8(\zeta'^2)/15]\tilde{\gamma}^2$ and $\gamma_{t,(100)} = (4\zeta'^2/15)\tilde{\gamma}^2$, respectively.

In the $\langle 111 \rangle$ direction, $\gamma_{l,(111)}^2 = [(5 + 20\zeta' + 20\zeta'^2 + 16\xi'^2)/45]\tilde{\gamma}^2$ and $\gamma_{t,(111)}^2 = [2(1 - \zeta'^2 + 2(\zeta'^2)/45)]\tilde{\gamma}^2$, whereas in the $\langle 110 \rangle$ direction, $\gamma_{l,(110)}^2 = [(2 + 6\zeta' + 7\zeta'^2 + 4\xi'^2)/15]\tilde{\gamma}^2$ and $\gamma_{t,(110)}^2 = (4\zeta'^2/15)\tilde{\gamma}^2$.

Therefore in our model the anisotropy of the glassy properties appears naturally. For more details, see Refs. [4, 9].
4.3. TRIGONAL LATTICE

Laermans and coworkers measured sound attenuation in quartz and observed also anisotropy (see [8] and references therein).

For a trigonal lattice of symmetry class $32$ (the symmetry of quartz), the matrix $[R]$ has the form [29]

$$[R] = \begin{pmatrix}
    r_{11} & r_{12} & r_{13} & r_{14} & 0 & 0 \\
    r_{12} & r_{11} & r_{13} & -r_{14} & 0 & 0 \\
    r_{13} & r_{13} & r_{33} & 0 & 0 & 0 \\
    r_{14} & -r_{14} & 0 & r_{14} & 0 & 0 \\
    0 & 0 & 0 & 0 & r_{44} & r_{14} \\
    0 & 0 & 0 & 0 & r_{14} & r_{11} - r_{12}^2 \\
\end{pmatrix}, \quad (12)$$

similar to that of the tensor of elastic stiffness constants, $[c]$, with $c_{ij}$ replaced by $r_{ij}$ [1, 32]. The system of coordinates that we use here is such that the $z$ and $x$ axes are the 3-fold and 2-fold rotational symmetry axes, respectively, while the $y$ axis is perpendicular to both $x$ and $z$. Solving the Christoffel equation we find that the crystal can sustain pure longitudinal waves propagating along the $x$ and $z$ axes, and pure transversal waves propagating along the $y$ and $z$ axes.

Calculating the transition rates, we obtain the anisotropic values of the $\gamma$ parameters,

$$\gamma_{k,k,l}^2 = \frac{2(r_{11}^2 + r_{12}^2 + r_{13}^2) + (r_{11} + r_{12} + r_{13})^2 + 4r_{14}^2}{15}, \quad (13)$$

$$\gamma_{k,k,l}^2 = \frac{8r_{13}^2 + 4r_{13}r_{33} + 3r_{33}^2}{15}, \quad (14)$$

$$\gamma_{k,y,t}^2 = \frac{4(r_{14}^2 + r_{44}^2)}{15}, \quad (15)$$

$$\gamma_{k,z,t}^2 = \frac{4(r_{14}^2 + r_{44}^2)}{15}. \quad (16)$$

Experiments on quartz have been done by Laermans and coworkers

4.4. HEXAGONAL LATTICE

The difference between the trigonal lattice of symmetry $32$ and the hexagonal lattice is that $r_{14}$ and $c_{14}$ are zero. This enhancement of symmetry allows propagation of pure longitudinal and transversal waves in all the three directions, $x$, $y$, and $z$. 
As before, we get the \( \gamma \) constants:

\[
\left( \gamma_{kk,\hat{t}} \right)^2 = \frac{2(r_{11}^2 + r_{12}^2 + r_{13}^2) + (r_{11} + r_{12} + r_{13})^2}{15} = \left( \gamma_{k\hat{x},\hat{t}} \right)^2,
\]

(17)

\[
\left( \gamma_{k\hat{y},\hat{t}} \right)^2 = \frac{8r_{13}^2 + 4r_{13}r_{33} + 3r_{33}^2}{15},
\]

(18)

\[
\left( \gamma_{k\hat{z},\hat{t}} \right)^2 = \left( \gamma_{k\hat{y},\hat{x}} \right)^2 = \frac{(r_{11} - r_{12})^2}{15},
\]

(19)

\[
\left( \gamma_{k\hat{x},\hat{z}} \right)^2 = \left( \gamma_{k\hat{y},\hat{x}} \right)^2 = \left( \gamma_{k\hat{x},\hat{z}} \right)^2 = \frac{4r_{44}^2}{15}.
\]

(20)

5. CONCLUSIONS

We reviewed briefly the main results we obtained in the description of the low temperature thermal properties of amorphous materials and disordered crystals. These materials contain dynamical defects whose contribution dominate the low temperature thermal properties.

We proposed a model for the two-level systems (TLS) which describe the dynamical defects and with this model we were able to calculate the thermal properties of dielectric nanoscopic membranes. Moreover, with our model we could explain the anisotropy observed in the glassy properties of disordered crystals. These peculiar anisotropies were observed experimentally by several authors, but were eventually never explained before [5–9].

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