This paper proposes an algorithm for the Lie symmetries investigation in the case of a high order evolution equation. General Lie operators are deduced and, in the next step, the associated conservation laws are derived. Due to the large number of equations derived from the symmetries conditions, the use of mathematical software is necessary, and we employed a MAPLE application. Some models arising from physics was chosen to test the method.

1. INTRODUCTION

Symmetry group methods and their recent generalizations have proved useful in understanding conservation laws, in constructing exact solutions and in establishing complete integrability of certain systems of differential equations [1–3].

In recent years considerable attention has been devoted to applications of symmetry group methods to a large variety of two or three order non-linear partial differential equations, but relatively few complete results have been obtained for the fourth order evolution equations. For physicists, the previous mentioned topics are very important in the study of concrete nonlinear dynamical systems with finite or infinite number of degrees of freedom. In the last case, various models of evolution equations for field theories have been analyzed from two main perspectives: (i) finding exact solutions and (ii) finding a coherent quantum description.

This paper present a simplified computational method to determines the Lie - Backlund generalized symmetries of a system of evolution equations and the associated conservation laws. The paper has the following structure: after this introductory part, a presentation of the computational methods for determining generalized symmetries and conservation laws will be given in the second section. The section 3 presents some results of the application of the methods to concrete physical equations.

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2. GENERALIZED SYMMETRIES AND CONSERVATION LAWS. METHODOLOGICAL APPROACH

2.1. GENERALIZED SYMMETRIES. DEFINITIONS

The symmetries encountered in field theories are usually of the type commonly referred to as point, or classical, symmetries. A point symmetry of a system of differential equations is a 1-parameter group of transformations of the underlying space of independent and dependent variables that carries any solution of the equations to another solution. For differential equations derived from a variational principle, the point symmetries which preserve the action lead to conservation laws. However, not all conservation laws stem from point symmetries. To account for all conservation laws in Lagrangian field theory one must enlarge the notion of symmetry to include classical and non-classical generalized symmetries.

A generalized symmetry is an infinitesimal transformation, constructed locally from the independent variables, the dependent variables, and the derivatives of the dependent variables, that carries solutions of the differential equations to nearby solutions. This kind of symmetries can be understood as generalized transformations which preserve the form of the equation. The importance of generalized symmetries is underlined by their role in completely integrable systems of non-linear differential equations. In particular, when a system of differential equations is integrable, it generally admits “hight orders” generalized symmetries Olver [3].

Let us consider a \( n \)-th order PDE system:

\[
\Delta_\nu(t, x, u^{(n)}[x]) = 0
\]

where \((t, x)\) represent the independent variables, while \( u \equiv \{u^\alpha, \alpha = 1, q\} \subset \mathbb{R}^q \) the dependent ones. The notation \( u^{(n)} \) designates the set of variables which includes \( u \) and the partial derivatives of \( u \) up to \( n \)-th order. The corresponding space is denoted by \( U^{(n)} \).

Let us consider the submanifold

\[
S_\Delta = \{(t, x, u^{(n)}) : \Delta_\nu(x, u^{(n)}) = 0\} \subset \chi \times U^{(n)}
\]

The set \( S_\Delta \) contains all the analytic solutions of the system (1). A symmetry group associated to the PDE system (1) consists in one-parameter group of transformations acting on an open subset \( M \subset \chi \times U \) which leave the set \( S_\Delta \) invariant:

\[
x^* = x + \varepsilon \xi(x, t, u) + O(\varepsilon^2),
\]

\[
t^* = t + \varepsilon \tau(x, t, u) + O(\varepsilon^2),
\]

\[
u^{\alpha,*} = u + \varepsilon \phi_\alpha(x, t, u) + O(\varepsilon^2), \quad \alpha = 1, q
\]

where \( \varepsilon \) is the group parameter, and \( \xi, \tau, \phi_\alpha \) are the infinitesimal generators of the symmetry group.
The general infinitesimal symmetry operator has the form:

$$U = \xi(t, x, u) \frac{\partial}{\partial x} + \tau(t, x, u) \frac{\partial}{\partial t} + \sum_{\alpha=1}^{q} \phi_\alpha(t, x, u) \frac{\partial}{\partial u^\alpha}$$  \hspace{1cm} (4)$$

where the notation $f[t, x, u]$ means a dependence of the function $f$ on the independent variables $t$ and $x$ and the dependent variables $u^\alpha, \alpha = 1, q$ with all their partial derivatives.

The $n$-th extension of (1), which is defined on the corresponding jet space $M^{(n)} \subset \chi \times U^{(n)}$, is given by:

$$pr^{(n)}(U) = U + \sum_{\alpha=1}^{q} \sum_{J} \phi^J_\alpha(t, x, u) \frac{\partial}{\partial u^J_\alpha},$$  \hspace{1cm} (5)$$

where

$$u^J_\alpha = \frac{\partial^m u^\alpha}{\partial^{j_1} t \partial^{j_2} x}.$$  \hspace{1cm} (6)$$

Also, in (5) the second summation refers to all the multi-indices $J = (j_1, j_2)$, with $1 \leq j_m \leq p, 1 \leq m \leq 2$. The coefficient functions $\phi^J_\alpha$ are given by the following formula:

$$\phi^J_\alpha(x, u^{(n)}) = D_J[\phi_\alpha - \xi u^\alpha_x - \tau u^\alpha_t] + \xi u^\alpha_{j_1 x} + \tau u^\alpha_{j_1 t}, \quad \alpha = 1, q$$  \hspace{1cm} (7)$$

in which

$$u^\alpha_x = \frac{\partial u^\alpha}{\partial x}, \quad u^\alpha_t = \frac{\partial u^\alpha}{\partial t}, \quad i = 1, p$$  \hspace{1cm} (8)$$

$$u^J_\alpha = \frac{\partial^m u^\alpha}{\partial^{j_1} t \partial^{j_2} x},$$  \hspace{1cm} (9)$$

$$D_J = D_{j_1}^{j_1} D_{j_2}^{j_2}.$$  \hspace{1cm} (10)$$

The Lie generalized symmetries for a PDE system which describes a dynamical system are the classical ones which keep invariant the differential system under a Lie group of local infinitesimal transformations. The Lie invariance condition is (Olver [3]):

$$pr^{(n)}(U)|\Delta|_{\Delta=0} = 0.$$  \hspace{1cm} (11)$$

The classical approach is to split the symmetries characteristic equation (11) by identifying the coefficients of all the corresponding monoms expressed in $u$ and his $x$-derivatives with zero. In practice, the equation (11) is very difficult to solve for an initial equation of order higher that two, due to the large numbers of terms involved in the expression of the prolongation.
2.2. GENERALIZED SYMMETRIES. COMPUTATIONAL APPROACH

An alternate version of the symmetries generating equation can be obtained by introducing the $q$-tuple \( Q = (Q^1, Q^2, \ldots, Q^q) \), known as the characteristic of the symmetry operator (4):

\[
Q^\alpha \equiv \phi_\alpha(t, x, u) - \xi [t, x, u] \frac{\partial u^\alpha}{\partial x} - \tau [t, x, u] \frac{\partial u^\alpha}{\partial t}, \alpha = 1, q
\]  

(12)

If the initial equation can be written as

\[
u_t^\alpha = K^\alpha[u]
\]

(where \( \alpha \) count the number of dependent variables \( \alpha = 1, q \)), then the symmetries determining equations (11) become:

\[
(D_t - K') (Q^1, \ldots, Q^q) = 0,
\]  

(13)

where \( D_t \) is the total time derivative (the evolutionary derivative)

\[
D_t Q^\alpha = \partial_t Q^\alpha + m \sum_{i=0}^q \sum_{\beta=1}^q \partial Q^\alpha \frac{\partial (u^\beta)^{(i)}}{\partial x} D_x^i (Q^\beta) = \partial_t Q^\alpha + (Q^\alpha)'(K)
\]

and the prime means the Fréchet derivative

\[
(K')_{\beta}^\alpha = \sum_{i=0}^{\infty} \frac{\partial K^\alpha}{\partial (u^\beta)^{(i)}} D_x^i.
\]

The simplest usual form of the symmetry generating equation is:

\[
\partial_t Q^\alpha + (Q^\alpha)'(K) = \sum_{\beta} (K')_{\beta}^\alpha (Q^\beta),
\]  

(14)

where \( \alpha, \beta = 1, q \).

Usually, one chooses an particular maximal order \( m \) for the characteristics

\[
Q^\alpha = Q^\alpha[t, x, u]
\]

and one searches for the solutions \( Q^\alpha \) of the equation (14) by identifying the coefficients of all the corresponding monoms expressed in \( w \) and his \( x \)-derivatives.

If the order of the studied dynamical system is \( n \) and the maximum order of characteristics is \( m \), the symmetry equation (14) becomes:

\[
\frac{\partial Q^\alpha}{\partial t} + \sum_{i=0}^m \sum_{\beta=1}^q \frac{\partial Q^\alpha}{\partial (u^\beta)^{(i)}} D_x^i (K^\beta) = \sum_{i=0}^n \sum_{\beta=1}^q \frac{\partial K^\alpha}{\partial (u^\beta)^{(i)}} D_x^i (Q^\beta)
\]  

(15)

for any \( \alpha = 1, q \).
We have to note that it is not at all compulsory that the maximum order \( m \) of derivatives which appear in \( Q \) should be the same with the order of the studied equation.

Due to the large number of equations derived from the symmetries conditions when the order of studied equation is higher that two, it must be useful to use a mathematical software in order to split (15) into a system of coefficients equations and to reduce this resulting system. We used a modified version of the JET package for MAPLE constructed by A. G. Meshkov (in [4]), but any other software package computing the total and the Fréchet derivative can be also used.

The infinitesimal generators of the symmetry operator \( \xi, \tau, \phi_\alpha \) are obtained by identifying \( \xi, \tau \) into the system

\[
Q^\alpha = \phi_\alpha - \xi u^\alpha_x - \tau K^\alpha, \alpha = 1, q.
\]  

(16)

2.3. CONSERVED DENSITIES AND CONSERVATION LAWS

A vector function \((\rho, \theta)\) on the jet space is called the conserved current for a system \( u_t = K(u) \) if it solves the equation

\[
D_t \rho = D_x \theta,
\]  

(17)

where \( D_t \) is the evolutionary derivative and \( D_x \) is the total derivative with respect to \( x \). The function \( \rho \) is said to be the conserved density and \( \theta \) is said to be the flux. The current \((D_x f, D_t f)\) is the conserved one for any system and it is called the trivial conserved current. If one add a trivial current to any other conserved current then the sum will be a conserved current also. The transformation \((\rho, \theta) \rightarrow (\rho + D f, \theta + D_t f)\) is called the equivalence transformation.

One can investigate the equation (17) with the help of the Euler operator \( E_\alpha \):

\[
E_\alpha = \sum_{n=0}^{\infty} (-D_x)^n \frac{\partial}{\partial (u^\alpha)_x^{(n)}},
\]  

(18)

where \((u^\alpha)_x^{(n)}\) is the \( n \) order spatial derivative of \( u \). The Euler operator \( E \) possesses an important property: \( Ef = 0 \) if and only if \( f = D_x(F) \) (see [3]). Applying the operator \( E \) to the equation (17) we obtain the following equation for the conserved densities:

\[
E_\alpha (D_t \rho) = E_\alpha \left( \frac{\partial \rho}{\partial t} + \sum_{i=0}^{m} \sum_{\beta=1}^{q} \frac{\partial \rho}{\partial (u^\beta)^{(i)}_x} D_x^i (u^\beta_t) \right) = 0.
\]  

(19)

The equations (19) can be solved by splitting it in many equations, following different order in the derivatives of the field variables \( u \). To do it we have to choose the maximum order of derivative involved in the expression of the density
\( \rho = \rho(t, x, u, u_x, u_{xx}, \ldots, u_x^{(m)}) \) and to use the same splitting procedure as in the case of the symmetries generating system.

Note that this method produces only the conserved densities from a conservation law. The corresponding flux is not determined. More, this method does not generate any association between symmetry operators and conservation laws, as in the case of Noether theorem (for Lagrangian systems). An alternative method, which doesn’t need any Lagrangian formalism, is based on an identity of Kara and Mahomed [5]. If

\[
U = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \sum_{\alpha=1}^{q} \phi_\alpha \frac{\partial}{\partial u^\alpha}
\]

is a symmetry operator for the system

\[
u_t^\alpha = K^\alpha[u],
\]

then the equation \( D_t \rho - D_x \theta = 0 \) is a conservation law invariant under \( U \) if and only if

\[
\begin{align}
prU(\rho) - \xi D_t \tau + \tau D_t \xi &= 0 \\
prU(\tau) - \tau D_x \xi + \xi D_x \tau &= 0.
\end{align}
\]  \( \text{(21)} \)

The same procedure of splitting this system by different order of the derivatives of the field variables \( u^\alpha \) is used to obtain associated conservation laws for any generalized symmetries.

3. APPLYING THE METHODOLOGY. EXAMPLES

We give three examples of PDE arising from physics used as test of the methods described before.

3.1. THE CALABI FLOW

Calabi flow equation is a fourth order differential equation obtained by the deformation of the second order Ricci flow equation. Both equations, Calabi flow and Ricci flow, represent versions of intrinsic flows describing the geometric evolution on a manifold with Riemannian metrics given by the Ricci curvature. The interest for these types of flow equations comes from their connection with the General Relativity, offering important tools in the study of the black holes and in the attempt of obtaining a quantum theory of gravity [6]. The Calabi Flow in local coordinate write as

\[
\partial_t u = -\partial_{xy} \left( \frac{1}{u} \partial_{xy} \ln(u) \right).
\]  \( \text{(22)} \)

In a previous paper, we considered a version of the Calabi flow in 1+1 dimensions, obtained in [7] from the local expression of Calabi flow in 2+1 dimensions...
(see [6]) by uni-directionalization procedure:

\[
    u_t = -\frac{u_{xxxx}}{u^2} + \frac{6u_{xxx}u_x}{u^3} - \frac{21u_{xx}u_x^2}{u^4} + \frac{4u_{xx}^2}{u^5} + \frac{12u_x^4}{u^5}.
\]  

(23)

Applying directly the method described in 2.2, one obtains that the group of all arbitrary-order generalized symmetries for the uni-directional Calabi flow equation in 1+1 dimensions is spanned by

\[ \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \text{ and } (x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t}), \]

which are geometrical symmetries.

On the other hand, the Calabi flow equation is supposed to be integrable [6] because it possesses a zero curvature representation and an infinite (non-standard) algebraic hierarchy of high order integrable equations. Indeed, when the first method described in (2.3) is employed, one obtains that for any set of arbitrary functions \( f_0, \ldots, f_k \) which verify

\[
    \partial_x \left[ f_0 - \frac{\partial f_1}{\partial x} + \frac{\partial^2 f_2}{\partial x^2} + \ldots + (-1)^n \frac{\partial^n f_n}{\partial x^n} \right] = \partial_t \left[ \sum_{k=0}^{n} (-1)^k \frac{\partial^k f_k}{\partial x^k} \right] = 0. 
\]  

(24)

The expression

\[
    \rho = \sum_{k=0}^{n} f_k D_x^k(u) 
\]  

(25)

is a conserved density for Calabi flow. The second method generates only the non-trivial conservation laws:

\[
\begin{align*}
    D_t h(u, t) u_x &= -D_x [h D_{xx} - \frac{1}{u} D_{xx} \ln(u)] - \int \frac{\partial h(u, t)}{\partial t} du, \\
    D_t x^2 u_x &= -D_x [x^2 \frac{1}{u} D_{xx} \ln(u) - 2x D_x \frac{1}{u} D_{xx} \ln(u)] + 2 \frac{1}{u} D_{xx} \ln(u), \\
    D_t x^2 u_x &= -D_x [x D_x \frac{1}{u} D_{xx} \ln(u) - \frac{1}{u} D_{xx} \ln(u)].
\end{align*}
\]  

(26)

where \( h \) is an arbitrary derivable function.

### 3.2. THE FOKKER-PLANCK EQUATION

The Fokker-Planck equation

\[
    u_t = (x u_x)_x + u_{xx}
\]  

(27)

describes the time evolution of the probability density function of the position of a particle in Brownian motion, inside a fluid.

Saccomandi [8] studied this equation under the form

\[
    v_t = xu + u_x v_x = u,
\]  

(28)
proving that this equation admits the non-local symmetry operator

\[ U = -x \frac{\partial}{\partial x} - \frac{\partial}{\partial t} + [u(x^2 + 2) + 2xv] \frac{\partial}{\partial u} + v(x^2 + 1) \frac{\partial}{\partial v}. \]  

(29)

We used this equation as a test system for the second method exposed in the section (2.3). The associated conservation law obtained is

\[ D_t \left[ \frac{1}{x} (e^{t - \ln(x)} x ve^{x^2/2}) \right] = D_x \left[ e^{t - \ln(x)} x^2 (u + xv) e^{x^2/2} \right]. \]  

(30)

3.3. THE BRETHERTON EQUATION

The Bretherton equation

\[ u_t = \frac{\partial}{\partial x} (u^3 u_{xxx}) \]  

(31)

is a classical free surface flow problem that describes the propagation of an air finger into a rigid-walled fluid-filled channel. This equation possesses self similar solution, travelling waves solutions and rational solutions (\cite{9}, chapter 10) but it is considered as a non-integrable equation.

The analysis of the existence of generalized symmetries with the method described in the section (2.2) gives the class of symmetry operators

\[ V_n = (-\frac{3}{4} xu_x^{(n)} + uu_x^{(n-1)}) \frac{\partial}{\partial u}. \]  

(32)

The only point symmetries of the Bretherton equation are

\[
\begin{cases}
V^x = \frac{\partial}{\partial x} \\
V^u = \frac{\partial}{\partial u} \\
V_1 = \frac{3}{4} x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}
\end{cases},
\]  

(33)

where the first one is a space translation, and the second one a translation along the orbits.

The study of the conservation laws of the Bretherton equation will be the object of a forthcoming paper.
REFERENCES