EIGEN VALUES OF SIX-DIMENSIONAL SYMMETRIC TENSORS OF CURVATURE AND WEYL TENSORS FOR $Z = (z-t)$-TYPE PLANE FRONTED WAVES IN PERES SPACE-TIME

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In this paper, we have shown the existence of eigen values of six-dimensional symmetric tensor obtained from curvature tensor, weyl conformal curvature tensor and projective curvature tensor respectively for plane fronted waves in peres space-time.

Key words: Plane fronted waves, metric tensor, Riemannian curvature tensor, Ricci tensor ($R_{ijkl}$), Weyl conformal curvature tensor ($C_{ijkl}$) and Projective curvature tensor ($W_{ijkl}$).

1. INTRODUCTION

The wave like exact solutions of the Einstein’s field equation are plane, spherical or cylindrical gravitational waves. The plane gravitational waves $g_{ij}(Z)$ are mathematically exposed by H. Takeno (1961), in general relativity Takeno [1] has deduced the line elements for $(z-t)$ and $(t/z)$- type waves as

$$ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - dz^2 + dt^2.$$  \hfill (1.1)

and

$$ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - Z^2(C-E)dz^2 - 2ZEdzdt + (C+E)dt^2$$ \hfill (1.2)

respectively.

Peres [2] deduced the space-time for $Z = (z-t)$ type plane fronted waves as

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - 2f(x,y,Z)(dz - dt)^2.$$ \hfill (1.3)

where $Z= (z-t)$ and $g_{ij} = g_{ij}(f), f = f(x,y,Z), Z = (z,t).$
The necessary and sufficient condition that space-time (1.3) satisfies field equation

\[ R_{ij} = 0 \text{ and } \Delta f = 0 \]

\[ \Delta = \partial_{11} + \partial_{22}, \partial_{11} = \frac{\partial^2}{\partial x \partial x}, \partial_{22} = \frac{\partial^2}{\partial y \partial y} . \]

Takeno (1961) denote (O) – system the original co-ordinate system in which metric is (1.3) and 0- system is a co-ordinate system in which the metric is of the form as (1.3) and calculate non vanishing components of \( R_{ijkl} \) and \( R_{ij} \) are as follows

\[ R_{a3b3} = -R_{a4b4} = R_{a4b4} = \partial_{ab} f \]

and

\[ R_{33} = -R_{34} = R_{44} = \Delta f . \quad (a,b = 1,2). \]

Deshmukh and Karade [3] have shown the existence of eigen values of six-dimensional symmetric tensor obtained from Riemannian curvature tensor \( R_{ijkl} \), from Weyl conformal curvature tensor \( C_{ijkl} \) and projective curvature tensor \( W_{ijkl} \)

for both the \((z – t)\) and \( \left( \begin{array}{c} t \\ z \end{array} \right) \) – type plane gravitational waves with the metric (1.1) and (1.2) respectively.

In this paper, we want to deduce the same for \( Z = (z – t) \) type plane fronted waves in peres space-time whose metric is (1.3).

2. EIGEN VALUES OF THE SIX-DIMENSIONAL SYMMETRIC TENSOR \( R_{\alpha\beta} = R_{ijkl} \) FOR \( Z = (z – t) \)- TYPE WAVES WITH METRIC (1.3)

Non-vanishing components of \( R_{ijkl} \) and \( R_{ij} \) from (1.3) are

\[ R_{1313} = -R_{1314} = R_{1414} = \partial_{11} f , \]

\[ R_{2323} = -R_{2324} = R_{2424} = \partial_{22} f , \]

\[ R_{1323} = -R_{1324} = -R_{1423} = R_{1424} = \partial_{12} f , \]

and

\[ R_{33} = -R_{34} = R_{44} = \Delta f . \]

To calculate non vanishing components of symmetric tensor \( R_{\alpha\beta} \) for (1.3), let renumbering the indices as \((12) = 1, \quad (13) = 2, \quad (14) = 3, \quad (23) = 4, \quad (25) = 5, \quad (34) = 6 \)

The components of \( R_{\alpha\beta} \) using (2.1) are

\[ R_{22} = R_{33} = -R_{23} = \partial_{11} f , \]

\[ R_{24} = -R_{25} = -R_{34} = R_{35} = \partial_{12} f , \]

\[ R_{45} = -R_{46} = R_{55} = \partial_{22} f . \]
Also non-vanishing independent component of $g_{a\beta}$ are
\[
\begin{align*}
g_{a\beta} &= g_{ijkl} = g_{i}\delta_{j\beta} - g_{i\alpha}g_{j\beta} \\
g_{22} &= g_{45} = 2f + 1 \\
g_{33} &= g_{55} = 2f - 1 \\
g_{45} &= g_{23} = -2f \\
g_{66} &= g_{11} = 1.
\end{align*}
\]

The eigen values of six-dimensional symmetric tensor $R_{a\beta}$ are given by determinantal equation
\[
R_{a\beta} - \lambda g_{a\beta} = 0,
\]

\[
\lambda^2 \begin{vmatrix}
\partial_{11} f - 2\lambda f - \lambda & -\partial_{11} f + 2\lambda f & \partial_{12} f & -\partial_{12} f \\
-\partial_{11} f + 2\lambda f & \partial_{11} f - 2\lambda f + \lambda & -\partial_{12} f & \partial_{12} f \\
\partial_{12} f & -\partial_{12} f & \partial_{22} f - 2\lambda f - \lambda & -\partial_{22} f + 2\lambda f \\
-\partial_{12} f & \partial_{12} f & -\partial_{12} f + 2\lambda f & \partial_{22} f - \lambda(2f - 1)
\end{vmatrix} = 0.
\]

After simplifying, the above equation becomes,
\[
\lambda^2 \left\{ \left[ \partial_{11} f - \lambda(2f + 1) \right] \left[ \partial_{11} f - \lambda(2f - 1) \right] + \left[ \partial_{11} f - 2\lambda f \right] \left[ 2\lambda f - \partial_{11} f \right] \right\} = 0
\]

\[
\Rightarrow \lambda^6 = 0.
\]

\Rightarrow Six-dimensional symmetric tensor $R_{a\beta}$ has six zero eigen values.

Hence existence of $(z - t)$-type plane fronted waves imply six zero eigen values of six-dimensional symmetric tensor of curvature tensor $R_{a\beta} \equiv R_{ijkl}$.

3. EIGEN VALUES OF SIX-DIMENSIONAL SYMMETRIC TENSOR $C_{a\beta} = C_{ijkl}$ FOR Z = $(z - t)$ TYPE WAVES FOR METRIC (1.3)

We have,
\[
C_{ijkl} = R_{ijkl} + \frac{1}{2} \left[ g_{ij} R_{k\ell} - g_{ij} R_{k\ell} + g_{kl} R_{i\ell} - g_{kl} R_{i\ell} \right] + \frac{R}{6} \left[ g_{ij} g_{kl} - g_{ij} g_{kl} \right].
\]

\[
\text{Now,} \quad R = g^{\alpha\beta} R_{\alpha\beta}
\]
\[
= g^{11} R_{11} + g^{12} R_{12} + g^{22} R_{22} + g^{33} R_{33} + g^{34} R_{34} + g^{44} R_{44}
\]
\[
= (-1 + 2f - 4f + 1 + 2f) \Delta f
\]
\[
= 0.
\]

The non-zero component of $R_{ij}$ for (1.3) are
\[
R_{33} = -R_{34} = R_{44} = \Delta f \quad [\text{Takeno (1961) (55.5)}].
\]
Also \[ g^{33} = -(1 - 2f), \quad g^{44} = (1 + 2f), \quad g^{34} = 2f \] (3.3)

The non-zero components of \( R_{ijkl} \) are

\[
\begin{align*}
R_{1313} &= -R_{1314} = R_{1414} = \partial_{11} f \\
R_{2323} &= -R_{2324} = R_{2424} = \partial_{22} f \\
R_{1323} &= -R_{1324} = -R_{1423} = R_{1424} = \partial_{12} f \\
\end{align*}
\] [by (55.4), Takeno (1961)]. (3.4)

Hence using the above results the non-zero components of \( C_{ijkl} \) are

\[
\begin{align*}
C_{1313} &= R_{1313} + \frac{1}{2} \left[ g_{11} R_{33} - g_{13} R_{13} + g_{33} R_{11} - g_{13} R_{13} \right] + 0 \\
&= \partial_{11} f + \frac{1}{2} \left[ g_{11} R_{33} - 0 + 0 - 0 \right] + 0 \\
&= \partial_{11} f + \frac{1}{2} (-1) \Delta f \\
&= \partial_{11} f - \frac{\Delta f}{2}. \\
\end{align*}
\]

\[
C_{1314} = R_{1314} + \frac{1}{2} \left[ g_{11} R_{34} - g_{14} R_{13} + g_{34} R_{11} - g_{14} R_{14} \right] \\
= -\partial_{11} f - \frac{1}{2} [-\Delta f] \\
= -\left( \partial_{11} f - \frac{\Delta f}{2} \right). \\
\]

Similarly, we get

\[
\begin{align*}
C_{1414} &= \partial_{11} f - \frac{\Delta f}{2}, \quad C_{2323} = \left( \partial_{22} f - \frac{\Delta f}{2} \right), \quad C_{1423} = -\partial_{12} f, \quad C_{2324} = -\left( \partial_{22} f - \frac{\Delta f}{2} \right), \\
C_{1323} &= \partial_{12} f, \quad C_{2424} = \left( \partial_{22} f - \frac{\Delta f}{2} \right), \quad C_{1324} = -\partial_{12} f, \quad C_{1424} = \partial_{12} f. \quad (3.5)
\end{align*}
\]

The non-vanishing components of \( C_{\alpha\beta} \) using (3.5) are given as,

\[
\begin{align*}
C_{22} &= -C_{23} = C_{33} = \left( \partial_{11} f - \frac{\Delta f}{2} \right), \\
C_{24} &= -C_{25} = -C_{34} = C_{35} = \partial_{12} f, \\
C_{44} &= -C_{45} = C_{55} = \left( \partial_{22} f - \frac{\Delta f}{2} \right). \quad (3.6)
\end{align*}
\]
Also non-vanishing components of \( g_{\alpha\beta} \) are given by (2.4).

The eigen values for the six-dimensional conformal tensor \( C_{\alpha\beta} \) are given by determinant equation \[ |C_{\alpha\beta} - \lambda g_{\alpha\beta}| = 0 \]

i.e.

\[
\lambda^2 \left[ \partial_{11} f - \frac{\Delta f}{2} - \lambda (2f - 1) \right] \partial_{22} f - \partial_{12} f \partial_{12} f - \partial_{12} f + \frac{\Delta f}{2} + 2\lambda f
\]

\[
\partial_{22} f - \partial_{22} f + \frac{\Delta f}{2} + 2\lambda f \partial_{12} f - \partial_{12} f - 2\lambda f + \lambda
\]

After simplifying the above equation becomes

\[ \lambda^6 = 0 \]

\[ \Rightarrow \text{Six-dimensional symmetric tensor } C_{\alpha\beta} \text{ has six zero eigen values.} \]

4. EIGEN VALUES OF SIX-DIMENSIONAL SYMMETRIC TENSOR \( W_{\alpha\beta} = W_{ijkl} \)

FOR \( Z = (z - t) \) - TYPE WAVES WITH METRIC (1.3)

Using \( W_{ijkl} = R_{ijkl} - \frac{1}{3} \left( g_{ij} R_{jk} - g_{ik} R_{ji} \right) \), the non-vanishing components of \( W_{ijkl} \) are,

\[
W_{1313} = R_{1313} - \frac{1}{3} \left( g_{13} R_{13} - g_{11} R_{33} \right)
\]

\[
= \partial_{11} f - \frac{1}{3} \left[ 0 - (-1)\Delta f \right]
= \partial_{11} f - \frac{\Delta f}{3}
\]

\[
W_{1314} = R_{1314} - \frac{1}{3} \left( g_{14} R_{13} - g_{11} R_{34} \right)
\]

\[
= -\partial_{11} f - \frac{1}{3} \left[ -\Delta f \right]
= -\left( \partial_{11} f - \frac{\Delta f}{3} \right)
\]

Similarly, other component of \( W_{\alpha\beta} \) are given as
The non-vanishing components of $W_{\alpha \beta}$ are given as,

$$
W_{22} = -W_{23} = W_{31} = \left( \partial_{11} f - \frac{\Delta f}{3} \right), \\
W_{24} = -W_{25} = W_{54} = \partial_{12} f, \\
W_{44} = -W_{45} = W_{55} = \left( \partial_{22} f - \frac{\Delta f}{3} \right).
$$

Also non-vanishing components of $g_{\alpha \beta}$ are given by (2.4).

Eigen values of the six-dimensional symmetric tensor $W_{\alpha \beta}$ are given by the determinant equation as,

$$
0 = \det \begin{vmatrix}
\partial_{11} f - \frac{\Delta f}{3} - \lambda (2f + 1) & -\partial_{11} f + \frac{\Delta f}{3} + 2\lambda f & \partial_{12} f & -\partial_{13} f \\
-\partial_{11} f + \frac{\Delta f}{3} + 2\lambda f & \partial_{11} f - \frac{\Delta f}{3} - \lambda (2f + \lambda) & -\partial_{12} f & \partial_{13} f \\
\partial_{12} f & -\partial_{12} f & \partial_{22} f - \frac{\Delta f}{3} - 2\lambda f - \lambda & -\partial_{23} f + \frac{\Delta f}{3} + 2\lambda f \\
-\partial_{13} f & \partial_{13} f & -\partial_{23} f + \frac{\Delta f}{3} + 2\lambda f & \partial_{22} f - \frac{\Delta f}{3} - 2\lambda f + \lambda
\end{vmatrix}
$$

Solving the above determinant equation we obtain

$$\lambda^6 = 0$$

Thus the six-dimensional symmetric tensor $W_{\alpha \beta}$ has six zero eigen values.

5. CONCLUSIONS

1. The existence of $Z = (z - t)$-type plane fronted waves in Peres space-time are guaranteed by the six zero eigen values of six-dimensional symmetric tensors of Riemannian curvature tensor, Weyl conformal curvature tensor and projective curvature tensor respectively.

2. This is geometrical aspect for conformation of $Z = (z - t)$-type waves as a ‘plane fronted gravitational waves’ and $g_{ij}$ of (1.3) posses ‘plane (wave-like)’ character.
3. Furthermore this is analogous to one obtain the same for Takeno’s space-time (1.1) and (1.2) as waves are ‘purely plane gravitational’ waves.

4. So geometrically both the space times of Peres and Takeno related to ‘plane wave-like’ and ‘plane wave’ solutions of Einstein field equations of General Relativity for $Z = (z – t)$-type waves.

REFERENCES

