We investigate cosmic string in Bianchi type I cosmological model in the presence of bulk viscosity. To get the deterministic solution of the field equations, a relation between metric coefficient is assumed and behavior of the model is reduced to second order non-linear differential equation. It is shown that this equation admits an infinite family of solutions. Some physical consequences from these results are also discussed.

Key words: string cosmology, bulk viscosity.

1. INTRODUCTION

At the very early stage of evolution of the universe, it is generally assumed that during the phase transition (as the universe passes through it critical temperature) the symmetry of the universe is broken spontaneously as predicted by grand unified theories (Everett [1]; Kibble [2,3]; Vilenkin [4]; Zel’dovich et al. [5, 6]). Moreover, the investigation of cosmic string and their physical properties near such a string has received wide attention because it is believed that cosmic string give rise to density perturbation which lead to formation of the galaxies (Zel’dovich [7]; Vilenkin [4]). These cosmic strings have an energy and couple to the gravitational field. Therefore, it is interesting to study gravitational effect which arises from string by using Einstein’s equation.

The general treatments of strings were initiated by Letelier [8, 9] and Satchel [10]. Letelier [8] has obtained the solutions of Einstein’s field equations for a cloud of string with spherical, plane and particular case of cylindrical symmetric. Einstein’s field equations for a cloud of massive string in Bianchi type I and Kantowski Sach’s space time have been solved by Letelier [9]. Afterwards, Krori et al. [11] and Wang [12] have discussed the solutions of Bianchi type II, VI, VIII and IX for a cloud of string. Tikekar and Patel [13] and Chakraborty [14] have presented the exact solution of the Bianchi type III and spherical symmetric cosmology respectively for a cloud of string.
On the other hand, cosmological models of a fluid with viscosity play a significant role in the study of evolution of universe. It is well known that at an early stage of the universe when neutrino decoupling occurs, the matter behaves like a viscous fluid. The coefficient of viscosity is known to decrease as the universe expands. Viscous fluid cosmological models in early universe have been widely discussed in the literature (Krani and Mukherjee [15]; Singh and Beesham [16]; Bali and Sharma [17]). Recently, Bali and Upadhyaya [18] have discussed Bianchi type III string cosmological models with bulk viscosity, where the constant coefficient of the bulk viscosity is considered. Yadav et al. [19] have studied some Bianchi type I viscous string cosmological model for a cloud of string with bulk viscosity and Wang ([12], [20], [21], [22]) have discussed LRS Bianchi type I and Bianchi type III model for a cloud string with bulk viscosity.

Recently, Yadav et al. [23] has investigated the integrability of the cosmic string in Bianchi type III space time in the presence of bulk viscous fluid by applying the new technique. Motivated the situation discussed above, in this paper we have focused upon the problem of establishing a formalism for studying a new integrability of cosmic string in Bianchi type I space time in the presence of bulk viscosity.

2. FIELD EQUATIONS

We consider Bianchi Type I string cosmology model with metric

\[ ds^2 = -dt^2 + a^2(dx^2 + b^2(dy^2 + dz^2)) \]  

where \( a \) and \( b \) are functions of \( t \) only. In a co-moving coordinate system we set \( u^i = u_i = (1,0,0,0) \). Also, \( x^i = (0,a^{-1},0,0) \).

The Einstein field equations for a cloud of string with bulk viscosity are

\[ R^i_j - \frac{1}{2} R g^i_j = \rho u_i u^j - \lambda x_i x^j - \xi \theta (u_i u^j + g^i_j) \]  

where \( \rho \) is the rest energy for cloud of strings with particles attached to them and \( \lambda \) is string tension density and \( \xi \) is the coefficient of viscous fluid and

\[ \rho = \rho_p + \lambda \]  

Here, \( \rho_p \) being the particle energy density \( u_i \) is the four velocities for a cloud of particle and \( x_i \) are the four vectors representing the strings direction, which essentially is the direction of anisotropy. Thus,

\[ u_i u^i = -1 = -x_i x^i \quad \text{and} \quad u_i x^i = 0 \]  

The non-vanishing components of the Einstein field equations are

\[ \frac{2}{a} \frac{\dot{a}}{b} + \frac{\dot{b}}{b^2} = \rho \]  
(5)

\[ \frac{2}{b} \frac{\dot{b}}{b^2} = \lambda + \xi \theta \]  
(6)

\[ \frac{2}{a} \frac{\dot{a}}{b} + \frac{\dot{b}}{b} + \frac{a b}{a b} = \xi \theta \]  
(7)

The physical quantities that are of importance in cosmology are Proper volume \( V \), expansion scalar \( \theta \), shear scalar \( \sigma^2 \) and have the following expressions for the metric

\[ V = a b^2 \]  
(8)

\[ \theta = \frac{a}{a} + 2 \frac{b}{b} \]  
(9)

\[ \sigma^2 = \frac{2}{3} \left( \frac{b}{b} - \frac{a}{a} \right)^2 \]  
(10)

2.1. SOLUTIONS OF FIELD EQUATIONS

Here, we have three field equations connecting five unknown quantities \( a, b, \rho, \lambda, \xi \). Therefore, in order to obtain exact solutions we must need two more relation connecting the unknown quantities. We assume the analytic relation

\[ b = ma^n \]  
(11)

Using equations (9) and (11) in (7), we obtain

\[ (n+1) \frac{\ddot{a}}{a} + n^2 \frac{\dot{a}^2}{a^2} = \xi (2n+1) \frac{\dot{a}}{a} \]  
(12)

Let us consider

\[ \dot{a} = f(a) \]  
(13)
Using (13) in (12), we get
\[
\frac{df}{da} + \left( \frac{n^2}{n+1} \right) f = \xi^{\left( \frac{2n+1}{n+1} \right)}
\]  
(14)

This on integration reduces to
\[
f = \xi^{\left( \frac{2n+1}{n^2+n+1} \right)} a + \frac{P}{a^{n+1}}
\]  
(15)

where P is an integrating constant. Integrating (15), we get
\[
a = \frac{1}{\xi^{k_4}} \left[ k_1 + k_2 e^{k_3 \xi} \right]^{k_4}
\]  
(16)

This gives
\[
b = \frac{m}{\xi^{k_5}} \left[ k_1 + k_3 e^{k_5 \xi} \right]^{k_5}
\]  
(17)

Where \( k_1 = -P(n^2 + n + 1) \), \( k_2 = \frac{S}{2n+1} \), \( k_3 = \frac{2n+1}{n+1} \), \( k_4 = \frac{n+1}{n^2+n+1} \), \( k_5 = \frac{n(n+1)}{n^2+n+1} \)

(18)

Hence the metric (1) reduces to the form
\[
ds^2 = -dt^2 + \left[ \frac{k_1 + k_2 e^{k_3 \xi}}{\xi} \right]^{2k_4} dx^2 + m^2 \left[ \frac{k_1 + k_2 e^{k_5 \xi}}{\xi} \right]^{2k_5} (dy^2 + dz^2)
\]  
(19)

Using the suitable transformation
\[
\frac{k_1 + k_2 e^{k_3 \xi}}{\xi} = L \frac{\sin(\xi \tau)}{\xi}; L^4 = X; mL^{k_5} (y + z) = Y + Z,
\]  
(20)

the metric (19) reduces to
\[
ds^2 = \left[ \frac{L \cos(\xi \tau)}{k_3(k_1 - L \sin(\xi \tau))} \right]^2 d\tau^2 + \left[ \frac{\sin(\xi \tau)}{\xi} \right]^{2k_4} dX^2 + m^2 \left[ \frac{\sin(\xi \tau)}{\xi} \right]^{2k_5} (dY^2 + dZ^2)
\]  
(21)

The rest energy \( \rho \), the string tension density \( \lambda \), the particle density \( \rho_p \), expansion \( \theta \), shear \( \sigma \) and proper volume \( V \) for the model (21) are given by
Some Bianchi type I string cosmological models

\[ \rho = k_3^2 k_5 (2k_4 + k_5) \left[ \xi - \frac{k_1 \xi}{L \sin(\xi \tau)} \right]^2 \]  \hspace{1cm} (22)

\[ \lambda = 3 k_3^2 k_5^2 \left[ \xi - \frac{k_1 \xi}{L \sin(\xi \tau)} \right]^2 + k_1 k_2 k_3 \left[ \frac{\xi^2}{L \sin(\xi \tau)} - \frac{k_1 \xi^2}{L^2 \sin^2(\xi \tau)} \right] - \xi k_5 (k_4 + 2k_5) \left( \xi - \frac{k_1 \xi}{L \sin(\xi \tau)} \right) \]  \hspace{1cm} (23)

\[ \rho_p = 2k_3^2 k_5 (k_4 - k_5) \left[ \xi - \frac{k_1 \xi}{L \sin(\xi \tau)} \right]^2 + \left[ \xi k_5 (k_4 + 2k_5) - k_1 k_2 k_3 \frac{\xi}{L \sin(\xi \tau)} \right] \left[ \xi - \frac{k_1 \xi}{L \sin(\xi \tau)} \right] \]  \hspace{1cm} (24)

\[ \theta = k_5 (k_4 + 2k_5) \left( \xi - \frac{k_1 \xi}{L \sin(\xi \tau)} \right) \]  \hspace{1cm} (25)

\[ \sigma^2 = \frac{2}{3} \left[ k_5 (k_4 - k_5) \left( \xi - \frac{k_1 \xi}{L \sin(\xi \tau)} \right) \right]^2 \]  \hspace{1cm} (26)

\[ V = m^2 \left[ \frac{L \sin(\xi \tau)}{\xi} \right]^{-k_1^2 + 2k_3} \]  \hspace{1cm} (27)

From (22) and (24), we observe that the energy conditions \( \rho \geq 0 \) and \( \rho_p \geq 0 \) are satisfied. Also, from (23), we observe that the string tension density \( \lambda > 0 \) provided

\[ 3k_3^2 k_5^2 \left( \xi - \frac{k_1 \xi}{L \sin(\xi \tau)} \right) + k_1 k_2 k_3 \left[ \frac{\xi^2}{L \sin(\xi \tau)} - \frac{k_1 \xi^2}{L^2 \sin^2(\xi \tau)} \right] \]

\[ > \xi k_5 (k_4 + 2k_5) \left( \xi - \frac{k_1 \xi}{L \sin(\xi \tau)} \right) \]

In the absence of bulk viscosity i.e. when \( \xi \to 0 \), the model (21) reduces to

\[ ds^2 = -\left( \frac{L}{k_1 k_5} \right)^2 d\tau^2 + \tau^{2k_4} dX^2 + m^2 \tau^{2k_3} (dY^2 + dZ^2) \]  \hspace{1cm} (28)

The physical parameter \( \rho, \lambda, \rho_p \) and the kinematical parameters \( \theta, \sigma^2, V \) for this model are
\[ \rho = \left( \frac{k_1 k_3}{L \tau} \right)^2 k_5 (2k_4 + k_5) \]  
\[ \lambda = \left( \frac{k_1 k_3}{L \tau} \right)^2 k_3 (3k_3 - 1) \]  
\[ \rho_p = \left[ \frac{k_1^2 k_3^3 k_4^2 (1 + k_4 - k_3)}{L^2 \tau^2} \right] \]  
\[ \sigma^2 = \frac{2}{3} \left[ \frac{k_1 k_4 (k_3 - k_4)}{L \tau} \right]^2 \]  
\[ \theta = \frac{k_1 k_4 (k_3 + 2k_5)}{L \tau} \]  
\[ V = m^2 \tau^{k_1 + 2k_3} \]

From (29), (30) and (31), we observe that the energy conditions \( \rho \geq 0, \lambda \geq 0, \rho_p \geq 0 \) are fulfilled.

### 2.3. OTHER MODEL

In general, \( \xi \) is not constant through out of the fluid, so that \( \xi \) cannot be taken always constant, especially when the universe is expanding. Since in general \( \xi \) depends on temperature \( T \) and pressure \( \rho \), it is reasonable to consider \( \xi \) as a function of the time \( t \).

In this case (12), after integration, lead to

\[ a = \left[ b_4 + k_4^{-1} \int h(t) dt \right]^{k_4} \]  

where

\[ h(t) = c e^{k_5 \int \xi(t) dt} \]

and \( b_4, c \) are constants of integration, therefore, we also obtain

\[ b = m \left[ b_4 + k_4^{-1} \int h(t) dt \right]^{k_5} \]

Hence, in this case the metric (1) reduces to
Some Bianchi type I string cosmological models

\[ ds^2 = -dt^2 + \left[ b_c + k_4^{-1} \int h(t)dt \right]^{2k_4} dr^2 + m^2 \left[ b_c + k_4^{-4} \int h(t)dt \right]^{2k_4} d\Omega^2 \]  

(38)

\[ \rho = n(n+2) \left[ \frac{h(t)}{b_c + k_4^{-1} \int h(t)dt} \right]^2 \]  

(39)

\[ \lambda = \left( 3n^2 - \frac{2n}{k_4} \right) h^2(t) + \left[ n h - \xi (2n+1) h(t) \right] \left[ b_c + k_4^{-1} \int h(t)dt \right] \]  

(40)

\[ \theta = \frac{(1+2n)h(t)}{b_c + k_4^{-1} \int h(t)dt} \]  

(41)

\[ \rho_p = \frac{[-2n^2 + \frac{2n}{k_4}] h^2(t) - \left[ n h - \xi (2n+1) h(t) \right] \left[ b_c + k_4^{-1} \int h(t)dt \right]}{[b_c + k_4^{-1} \int h(t)dt]^2} \]  

(42)

\[ \sigma = \frac{\sqrt{2}}{\sqrt{3}} \left[ \frac{(n-1)h(t)}{b_c + k_4^{-1} \int h(t)dt} \right] \]  

(43)

\[ \frac{\sigma}{\theta} = \frac{\sqrt{2}}{\sqrt{3}} \frac{n-1}{2n+1} \]  

(44)

3. CONCLUSION

In this paper we have presented a Kantowski Sachs string cosmological model in the presence and absence of bulk viscosity. The model in the bulk viscosity is given by (21) represents an expanding universe when \( \sin(\xi \tau) > \frac{k_1}{L} \) and when \( \sin(\xi \tau) < \frac{k_1}{L} \) then \( \theta \) decreases with time. Therefore, the model describes a sharing non-rotating expanding universe without the big bang star. From the above discussion we can see that the bulk viscosity plays a significant role in the evolution of the universe. Since \( \lim_{\tau \to \infty} \frac{\sigma}{\theta} \neq 0 \), the model does not approach isotropy
for large value of \( \tau \). However, if \( \sin(\xi \tau) = \frac{k_1}{L} \). The model (23) represents isotropy model in the presence of bulk viscosity.

In the absence of bulk viscosity, the model (28) starts expanding with a big bang at \( \tau = 0 \) and the expansion in the model decreases as time increases when \( L \neq 0 \). Also, when \( \tau \to \infty \), the shear is zero. The physical parameter \( \rho, \lambda, \rho_p \) are infinite if \( k_2 \) is negative and when \( \tau = 0 \). Since \( \lim_{\tau \to \infty} \sigma \neq 0 \), the model does not approaches isotropy for large value of \( \tau \).

In the discussion of the model the equation (36) has a rich structure and admit different choice of structure \( \xi(t) \). We have to choose \( \xi(t) \) in such a manner so that (36) is integrable. We can choose \( \xi(t) \) the suitable exponential, polynomial and sinusoidal of the function \( \xi(t) \) such that (36) becomes integrable. So, we conclude that bulk viscosity play significant role in the evolution of the universe. In the presence of bulk viscosity model represent an expanding, shearing and non-rotating universe without the big bang start. But, in the absence of viscosity, the model start expanding with a big bang at \( \tau = 0 \).

REFERENCE