THE INVERSE PROBLEM IN SCATTERING THEORY FOR OPTICAL FIELD PROPAGATION IN INHOMOGENEOUS MEDIA WITH DIELECTRIC PERMITTIVITY VARIATION*

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Received November 4, 2009

In the paper was presented a scattering model of electromagnetic field using an inverse problem in scattering theory, which consists in data processing caused by optical scattering effect. Also, we presented the spectral theory of operators (linear and nonlinear) with applications in inverse scattering theory. The problem is interesting because it can be solved a nonlinear problem using a spectral data set of linear problems.

In scattering theory, the inverse problem consists in determination of potential function knowing spectral data for field amplitude and phase. The nonhomogenous scalar field can be described by nonhomogenous dielectric permittivity.

Key words: photonic network, nonhomogeneous dielectric permittivity.

INTRODUCTION

In [1-2] are presented models regarding inverse problem in scattering theory. The inverse problem [3] consists in spectral processing of data that cause the scattering. In [3] also, wave equations for axial components of electromagnetic field are solved. In [4], equation of radial solution $\varphi(r)$ comes to integral form proposed by Regge [5]. In radial field component, the $V(r)$ potential is proportional with $\varepsilon(r)$ – the inverse problem.

Solving the inverse problem in scattering theory using R. Newton method, consist in solving radial equation [6]. Eigenvalue equation is build and spectral data function $F(r;\rho)$ is estimated.

In the paper was presented spectral theory of nonlinear operators with applications in inverse scattering theory. Also, was presented a general model that use inverse scattering. In the paper, we have used spectral theory of nonlinear operators with applications in inverse scattering.

* Paper presented at the "Optoelectronic Techniques for Environmental Monitoring" (OTEM-2009), September 30–October 2, 2009, Bucharest, Romania.

We have studied analytical solutions for optical field propagation equations are obtained. The optical field polarizing state is described using Stokes parameters obtained by calculation of trajectories on Poincaré sphere.

By solving plane waves equation we obtained the expression:

$$e^{ikr} = \frac{4\pi}{kr} \sum_{l,m} i^l \cdot u_l(kr) \cdot y_l^m(\hat{\vec{r}}) y_l^{m*}(\hat{\vec{k}}), \tag{1}$$

which represent harmonic plain wave development by Riccati-Bessel functions in media with nonhomogeneous propagation of refractive index.
Let the Maxwell equations in a non-magnetic environment with no currents and no charges, with $\varepsilon = \varepsilon(t, \vec{r})$ dielectric permittivity. The Maxwell equations can be written in Gauss referential system:

$$\begin{aligned} &\vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial}{\partial t}(\varepsilon \vec{E}) \\ &\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{H}) \\ &\vec{\nabla}\vec{H} = 0 \\ &\vec{\nabla}(\varepsilon \vec{E}) = 0 \end{aligned} \tag{2}$$

Due to the non-homogeneity of dielectric permittivity ε, last equation from (2) can be written as:

$$\vec{\nabla}\vec{E} = -\vec{\nabla} \ln \varepsilon(\vec{r}, t) \cdot \vec{E} \tag{3}$$

Wave equations for electric and magnetic field intensity become:

$$\begin{aligned} &-\Delta\vec{H} = \frac{1}{c} \cdot \frac{\partial}{\partial t}(\vec{\nabla} \times \varepsilon \vec{E}) \\ &-\Delta\vec{E} = \vec{\nabla}(\vec{\nabla} \ln \varepsilon \cdot \vec{E}) - \frac{1}{c} \cdot \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H}) \end{aligned} \tag{4}$$

or developed as:

$$\begin{aligned} &\Delta\vec{E}(\vec{r}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\varepsilon(\vec{r}, t)\vec{E}) + \vec{\nabla}(\vec{\nabla} \ln \varepsilon \vec{E}) = 0 \\ &\Delta\vec{H}(\vec{r}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\varepsilon \vec{H}) + \frac{\partial^2 \varepsilon}{c^2 \partial t^2}\vec{H} + \frac{\partial \varepsilon}{c \partial t}\frac{\partial \vec{H}}{C \partial t} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\varepsilon \times \vec{E}) = 0 \end{aligned} \tag{5}$$

We select the axial components of field intensities as follows:

$$\begin{aligned} &\vec{\mathrm{E}} \to \vec{\mathrm{E}}(\mathrm{x}, \mathrm{y}) \cdot \mathrm{e}^{-\gamma z + \mathrm{i}\omega t} \\ &\vec{\mathrm{H}} \to \vec{\mathrm{H}}(\mathrm{x}, \mathrm{y}) \cdot \mathrm{e}^{-\gamma z + \mathrm{i}\omega t} \end{aligned} \tag{6}$$

where γ is the axial wave number and spatial non-homogenous dielectric permittivity become:

$$\varepsilon = \varepsilon(x, y, z) \tag{7}$$

In these circumstances, the wave equations (5) can be rewritten as:

$$\begin{aligned} &\Delta_T \vec{E}(x, y) + \left(\gamma^2 + \frac{\omega^2}{c^2}\varepsilon(x, y, z)\right)\vec{E}(x, y) + \vec{\nabla}_T\left[\vec{\nabla}_T \ln\varepsilon(x, y, z)\vec{E}(x, y)\right] = 0 \\ &\Delta_T \vec{H}(x, y) + \left(\gamma^2 + \frac{\omega^2}{c^2}\varepsilon(x, y, z)\right)\vec{H}(x, y) + \left(\frac{\mathrm{i}\omega}{c}\right)\left(\vec{\nabla}_T\varepsilon(x, y, z)\times\vec{E}(x, y)\right) = 0 \end{aligned} \tag{8}$$

where

$$k^2 = \gamma^2 + \frac{\omega^2}{c^2}\varepsilon(x, y, z) \tag{9}$$

represent total plane wave number of a nonhomogeneous media.

We choose axial component as:

$$\begin{aligned} &\vec{\mathrm{E}} \to \vec{k}E_z(x,y) = \vec{k}\Phi(x,y) \\ &\vec{\mathrm{H}} \to \vec{k}H_z(x,y) = \vec{k}\Phi(x,y) \end{aligned} \tag{10}$$

The wave equation for $\Phi(x,y)$ scalar field become:

$$\Delta_T\Phi(x, y) + \gamma^2\Phi(x, y) + \left[\frac{\omega^2}{c^2}\varepsilon(x, y, z) - \gamma\frac{\partial\ln\varepsilon(x, y, z)}{\partial z}\right]\Phi(x, y) = 0 \tag{11}$$

We consider a uniform travel in z direction to simplify solving (11) equation. We can write:

$$\begin{aligned} &\varepsilon = \varepsilon(x, y) \\ &\frac{\partial\varepsilon}{\partial z} = 0 \end{aligned} \tag{12}$$

The integral equation, in real space, has the expression:

$$\Phi(\rho) = \int_{-\infty}^{+\infty} H(\rho, \rho')\cdot\Phi(\rho')\,d\rho' \tag{13}$$

which is an integral equation Huygens-Fresnel type.

The integral nucleus of equation becomes:

$$H(\rho,\rho') = \frac{\omega^2}{c^2}\varepsilon(\rho')\cdot\frac{\sin\left[\gamma(\rho-\rho')\right]}{\gamma} \tag{14}$$

Also, the integral equation becomes:

$$\Phi(\rho) = \frac{\omega^2}{c^2}\int_{-\infty}^{+\infty}\varepsilon(\rho')\frac{\sin\left[\gamma(\rho-\rho')\right]}{\gamma}\Phi(\rho')d\rho' \tag{15}$$

Due to existence of stability and steady state conditions, we chose as media a photonic network $\varepsilon(\rho')$ which represents a channel sequence with hyperbolic cross-cutting section perpendicular on direction of propagation (fig.1). Thus, we can write:

$$\varepsilon(\rho') = \sum_{j=0}^{N}\frac{\delta\left(\rho-\rho'-\rho'_j\right)}{\mathrm{ch}\left(\rho-\rho'-\rho'_j\right)}\cdot\varepsilon', \tag{16}$$

where δ is Dirac function and ε' a calibration constant.

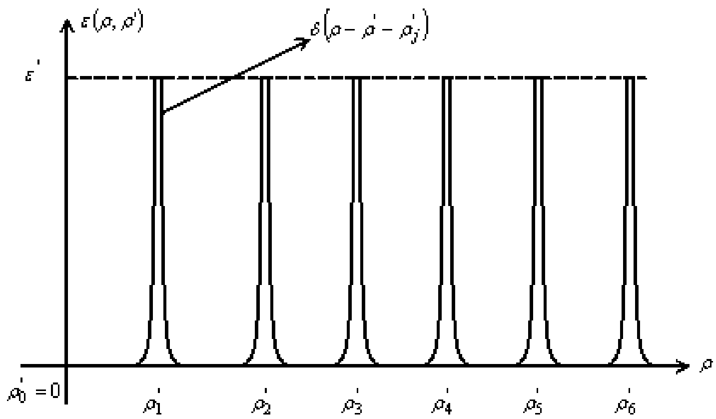

Fig.1 – Sequence of pulses induced by periodic variations of dielectric permittivity in a photonic network.

The integral wave equation becomes:

$$\Phi_N(\rho) = \frac{\omega^2}{c^2}\cdot\varepsilon'\cdot\sum_{j=0}^{N}\int_{-\infty}^{+\infty}\frac{\delta\left(\rho-\rho'-\rho'_j\right)}{\mathrm{ch}\left(\rho-\rho'-\rho'_j\right)}\cdot\frac{\sin\left[\gamma(\rho-\rho')\right]}{\gamma}\Phi_j(\rho')d\rho' \tag{17}$$

Processing the integral from (17), we obtain an algebraic equation:

$$\Phi_N(\rho) = \frac{\omega^2}{c^2} \cdot \varepsilon' \cdot \sum_{j=0}^{N} \left[\frac{\sin\left[\gamma\left(\rho - \rho'_j\right)\right]}{\gamma \cdot \mathrm{ch}\left(\rho - \rho'_j\right)} \right] \cdot \Phi_j\left(\rho'_j\right) \tag{18}$$

and for the case ρ=0

$$\Phi_N(0) = \frac{\omega^2}{c^2} \cdot \varepsilon' \cdot \sum_{j=0}^{N} \left[-\frac{\sin\left(\gamma \cdot \rho'_j\right)}{\gamma \cdot \mathrm{ch}\left(\rho'_j\right)} \right] \cdot \Phi_j\left(\rho'_j\right) \tag{19}$$

The wave equation solution of photonic network has expression:

$$\Phi_N(\rho) - \Phi_N(0) = \frac{\omega^2}{c^2} \cdot \varepsilon' \cdot \sum_{j=0}^{N} \Phi_j\left(\rho'_j\right) \cdot \left[\frac{\sin\left[\gamma\left(\rho - \rho'_j\right)\right]}{\gamma \cdot \mathrm{ch}\left(\rho - \rho'_j\right)} + \frac{\sin\left(\gamma \cdot \rho'_j\right)}{\gamma \cdot \mathrm{ch}\left(\rho'_j\right)} \right] \tag{20}$$

Theory

Let consider a calculation example for radial component of $\varphi(r)$ field resulted from solution of wave equation in spherical coordinates:

$$\Phi(r, \theta, \varphi) = \frac{\varphi(r)}{\sqrt{r}} Y_m^{\sigma}(\theta, \varphi) \tag{21}$$

The equation of radial solution $\varphi(r)$ has the exact form:

$$\frac{d^2\varphi(r)}{dr^2} + \frac{1}{r}\frac{d\varphi(r)}{dr} + k^2\varphi(r) - \frac{\lambda^2}{r^2}\varphi(r) - V(\vec{r})\varphi(r) = 0 \tag{22}$$

where: $\lambda = \sigma + \frac{1}{2}$, with natural integer σ.

The solution of radial component of optical field can be written as:

$$\varphi(r) = \varphi(r/k, \lambda) \tag{23}$$

The radial asymptotic solution for $\varphi(r)$ has the expression:

$$\lim_{r\to\infty} \varphi(r/k, \lambda) = \sqrt{\frac{2}{\pi r}} \cdot A(k, \lambda) \cdot \sin\left[k \cdot r - \frac{\pi}{2}\left(\lambda - \frac{1}{2}\right) + \delta(k, \lambda) \right], \tag{24}$$

where $A(k,\lambda)$ represent the amplitude of 'scattered' wave on $V(r)$ potential function, and $\delta(k,\lambda)$ phase of the scattered wave.

In scattering theory, the inverse problem consist in determination of $V(r)$ by knowing the spectral data $A(k,\lambda)$ and $\delta(k,\lambda)$. In order to simplify calculation, we study further the case:

$$k = 1 \tag{25}$$

For this particular case, the spectral data has the following form:

$$\begin{aligned} A(1,\lambda) &= A(\lambda)\big|_{\lambda=\sigma+1/2} \\ \delta(1,\lambda) &= \delta(\lambda)\big|_{\lambda=\sigma+1/2} \end{aligned}. \tag{26}$$

Taking into account (25), the wave equation has the expression:

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} + \varphi - \frac{\lambda^2}{r^2}\cdot\varphi - V(r)\cdot\varphi = 0\Big|_{V(r)=-\frac{\omega^2}{c^2}\varepsilon(r)} \tag{27}$$

With the purpose of emphasizes spectral amplitude and phase, Regge [5] uses an integral form with eigenvalues for radial equation:

$$\varphi(r,\lambda) = I_\lambda(r) + \int_0^r K(r,\rho)\cdot I_\lambda(\rho)\frac{d\rho}{\rho}, \tag{28}$$

where $I_\lambda(r)$ is first kind λ order Bessel function. We use expression of potential function $V(r)$ under the form:

$$V(r) = -\frac{\omega^2}{c^2}\varepsilon(r) \tag{29}$$

where

$$\varepsilon(r) = \varepsilon'\cdot\sum_{\sigma'=0}^{\infty} C_{\sigma'+\frac{1}{2}} I_{\sigma'+\frac{1}{2}}(r)\frac{1}{r} \tag{30}$$

We can calculate the radial solution of the field:

$$\varphi\left(r,\sigma+\frac{1}{2}\right) = I_{\sigma+\frac{1}{2}}(r) - \frac{1}{2}\int_0^r V(\rho)\cdot I_{\sigma+\frac{1}{2}}(\rho)\,d\rho \tag{31}$$

From (30)-(31) equations result an algebraic equation system defined as:

$$\varphi\left(r,\sigma+\frac{1}{2}\right) = I_{\sigma+\frac{1}{2}}(r) + M_{00}\cdot\sum_{\sigma'=0}^{\infty} C_{\sigma'+\frac{1}{2}} M_{\sigma,\sigma'}(r) \tag{32}$$

where

$$M_{00} = \frac{1}{2}\cdot\frac{\omega^2}{c^2}\cdot\varepsilon' \text{ and } M_{\sigma,\sigma'}(r) = \int_0^r I_{\sigma'+\frac{1}{2}}(\rho) I_{\sigma+\frac{1}{2}}(\rho)\frac{d\rho}{\rho} \tag{33}$$

From (32) algebraic equation system $C_{\sigma'+\frac{1}{2}}$ coefficients can be calculated. Also, from (30), the function $\varepsilon(r)$ can be calculated in terms of variable coefficients as in inverse problem. The $V(r)$ function is introduced in (31) and we calculate radial component of the field. Further we can calculate:

$$I_{1/2}(r)=\sqrt{\frac{2}{\pi\cdot r}}\mathrm{sh}(r)$$
$$I_{3/2}(r)=\sqrt{\frac{2}{\pi\cdot r}}\mathrm{ch}(r) \tag{34}$$

$$M_{0,0}=\frac{2}{\pi}\left[\frac{1-\mathrm{ch}(2r)}{4r}-\mathrm{sh}i(2r)\right]$$
$$M_{1,0}=\left(\frac{r^{\delta-1}}{\delta-1}\right){}_1F_2\left(\frac{\delta-1}{2};\frac{\delta-1}{2}+1;\frac{1}{2}+\delta;\frac{r^2}{4}\right) \tag{35}$$
$$M_{1,1}=M_{0,0}$$

$$C_{1/2}(r)=\frac{\varphi(r,1/2)-I_{1/2}(r)}{M_{0,0}\left(\frac{2}{\pi}\right)\left(\frac{1-\mathrm{ch}(2r)}{2r}-\mathrm{sh}i(2r)\right)}$$
$$C_{3/2}(r)=\frac{\left[\varphi(r,3/2)-I_{3/2}(r)\right]M_{0,0}-\left[\varphi(r,1/2)-I_{1/2}(r)\right]M_{1,0}(r)}{M_{0,0}\frac{4}{\pi^2}\left[\frac{1-\mathrm{ch}(2r)}{2r}-\mathrm{sh}i(2r)\right]^2} \tag{36}$$

We calculate:

$$\varepsilon(r)=\frac{\varepsilon'}{r}\cdot\left[C_{1/2}(r)I_{1/2}(r)+C_{3/2}(r)I_{3/2}(r)\right]$$
$$V(r)=-\frac{\omega^2}{c^2}\varepsilon(r) \tag{37}$$

The solution of radial equation for a photonic network is:

$$\varphi(r,1/2)=I_{1/2}(r)-\frac{1}{2}\int_0^r V(r')I_{1/2}(r')\,\mathrm{d}r'$$
$$\varphi(r,3/2)=I_{3/2}(r)-\frac{1}{2}\int_o^r V(\rho)I_{3/2}(\rho)\,\mathrm{d}\rho \tag{38}$$

RESULTS AND DISCUSSIONS

We purchase in solving the inverse problem in scattering theory using a method that imply eigenvector and eigenvalue.
For equation of radial solution φ(r):

$$\frac{\mathrm{d}^2\varphi(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\varphi(r)}{\mathrm{d}r}+\varphi(r)-\frac{\lambda^2}{r^2}\varphi(r)-V(r)\varphi(r)=0\,. \tag{39}$$

We attach integral equation:

$$\varphi(r,\lambda)=I_\lambda(r)+\int_0^r K(r,\rho)I_\lambda(\rho)\frac{\mathrm{d}\rho}{\rho} \tag{40}$$

We build eigenvalue equation:

$$\hat{D}(r)K(r,\rho)=\hat{D}_o(\rho)K(r,\rho) \tag{41}$$

where

$$\begin{aligned}\hat{D}(r)&=r^2\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+1-V(r)\right]\\ \hat{D}_0(\rho)&=\rho^2\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}+\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}+1\right]\end{aligned} \tag{42}$$

are the operators of eigenvalue problem.
The potential function result:

$$V(r)=\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}K(r,r) \tag{43}$$

The Regge method [5] consists in solving eigenvalue equation:

$$\hat{D}_o(r)F(r,\rho)=\hat{\Delta}_o(\rho)F(r,\rho) \tag{44}$$

with initial condition:

$$F(0,\rho)=F(r,0)=0 \tag{45}$$

where $F(r,\rho)$ is a *spectral data function*.

Using decomposition after eigenvalues and eigenfunctions, this function can be written as:

$$F(r,\rho)=\sum_{\sigma=0}^{\infty}C_{\sigma+\frac{1}{2}}I_{\sigma+\frac{1}{2}}(r)I_{\sigma+\frac{1}{2}}(\rho) \tag{46}$$

Analogous, the equation for integral nucleus $K(r,\rho)$ can be obtained in terms of $F(r,\rho)$:

$$K(r,\rho) = F(r,\rho) + \int_0^r K(r,z) F(z,\rho) \frac{\mathrm{d}z}{z} \tag{47}$$

Also $K(r,\rho)$ is function of $\varphi(r,\sigma+1/2)$:

$$K(r,\rho) = \sum_{\sigma=0}^{\infty} C_{\sigma+\frac{1}{2}} I_{\sigma+\frac{1}{2}}(\rho)\, \varphi\left(r, \sigma+\frac{1}{2}\right) \tag{48}$$

The potential function for a photonic network is obtained as:

$$V(r) = \left(\frac{2}{r}\right) \sum_{\sigma=0}^{\infty} C_{\sigma+\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}r}\left[\varphi\left(r,\sigma+\frac{1}{2}\right) I_{\sigma+\frac{1}{2}}(r)\right] \tag{49}$$

where

$$\varphi\left(r,\sigma+\frac{1}{2}\right) = I_{\sigma+\frac{1}{2}}(r) + \sum_{\sigma'=0}^{\infty} C_{\sigma'+\frac{1}{2}}\, \varphi\left(r,\sigma'+\frac{1}{2}\right) M_{\sigma,\sigma'}(r) \tag{50}$$

and

$$M_{\sigma,\sigma'}(r) = \int_0^r I_{\sigma+\frac{1}{2}}(\rho)\, I_{\sigma'+\frac{1}{2}}(\rho) \frac{\mathrm{d}\rho}{\rho} \tag{51}$$

CONCLUSIONS

In the paper, we solved the inverse problem in scattering theory in media characterized by nonhomogenous variation of dielectric permittivity.

Using Maxwell equations and by selecting the axial components of electromagnetic fields, we determined the integral solution of field equation. We obtained wave equation solution for a photonic network (20).

Starting from wave equation solution in spherical coordinates and considering eigenvalue model proposed by Regge [5], we obtained radial equation solution for a photonic network.

Using a method that implies eigenvectors and eigenvalues equations, we were able to solve the inverse problem in scattering theory. The potential functions with radial geometry were calculated in an eigenvalue problem in connection with the inverse problem.

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