FRACTIONAL DIMENSIONAL HARMONIC OSCILLATOR

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The fractional Schrödinger equation corresponding to the fractional oscillator was investigated. The regular singular points and the exact solutions of the corresponding radial Schrödinger equation were reported.

Key words: fractional space, fractional oscillator, Schrödinger equation.

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1. INTRODUCTION

Fractional dimensional space has successfully been applied as an effective physical description of confinement in low-dimensional system. This method replaces the real confining structure with an effective space, where the measure of the anisotropy or confinement is given by the non-integer dimension (see for example [1]-[4] and the references therein).

As it is known, many of the investigations into low-dimensional semiconductors structures are based on a mathematical basis introduced in [5] in which he described integration on a space of \(\alpha\) dimensions and provided a generalization of the Laplace operator on this space.

Recent progress includes the description of a single coordinate momentum operator in this fractional dimensional space based on generalized Wigner relations [6,7] presenting realization of parastatics [8].

The fractional dimensions appears as an explicit parameter when the physical problem is formulated in \(\alpha\) dimensions such that \(\alpha\) may extended to non-integer values, as in Wilson’s study of quantum field theory models in less than four dimen-

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sions [9], or in the approach to quantum mechanics mentioned in [5]. On the other hand the experimental measurement of the dimensional $\alpha$ of our real world is given by $\alpha = (3 \pm 10^{-6})$ [5, 9]. The fractional value of $\alpha$ agrees with the experimental physical observations that in general relativity, gravitational fields are understood to be geometric perturbations (curvatures) in our space-time [10], rather than entities residing within a flat space-time. On the other hand, Zeilinger and Svozil [11] reported that the current discrepancy between theoretical and experimental values of the anomalous magnetic moment of the electron could be resolved if the dimensionality of space $\alpha$ is $\alpha = 3 - (5.3 \pm 2.5) \times 10^{-7}$.

The simple harmonic oscillator is one of the most important topics in quantum mechanics. It is applicable in different physical situations and has the great advantage that it has closed solutions for the energy eigenvalues and eigenfunctions [12]. In the context of connecting the Coulomb and oscillator potential in arbitrary dimensions obtain the generalized KS transformation: from five-dimensional hydrogen to eight-dimensional isotropic oscillator [13].

The system of a fractional-dimensional Bose-like oscillator of one degree of freedom whose canonical variables satisfy the general Wigner commutation relations and the momentum-position uncertainty relations are obtained in a fractional-dimensional space [14]. Besides in [15], the authors formulated an algebra approach to quantum mechanics in fractional dimensions in which the momentum and the position operators satisfy the $R$-deformed Heisenberg relations which lead to the paraboson operators which are used to solve harmonic oscillator in $d$ dimensional fractional space. The main aim of this paper is to find the exact solutions of the Schrödinger equation of the fractional dimensional oscillator. The plan of the paper is as follows: In Section 2 the model is presented. In Section 3 we have presented the exact solutions corresponding to several values of $\beta$. Finally, Section 4 presents our conclusions.

2. THE MODEL

The Schrödinger equation to start with is given below

$$\left(\frac{\hbar^2}{2\mu r^{\alpha - 1}} \frac{\partial}{\partial r} (r^{\alpha - 1} \frac{\partial}{\partial r}) + \frac{\Lambda_\alpha^2(\Omega)}{2\mu r^2} + \frac{1}{2} k_\beta r^{\beta - 1}\right) \Psi(r, \Omega) = E \Psi(r, \Omega),$$

(2.1)

where $\Lambda_\alpha^2(\Omega) \Psi(r, \Omega) = -l(l + \alpha - 2) \Psi(r, \Omega)$, for $l = 0, 1, 2, 3$, $k_\beta$ is the spring constant with units $\text{Joule} / \text{(meter)}^{\beta - 1}$, $\alpha$ and $\beta$ are dimensional fractional space.

The next step is to make use of the change of variable $\Psi(r, \Omega) = R(r) Y(\Omega)$. 
As a result the radial equation part becomes
\[ R''(r) + \frac{\alpha - 1}{r} R'(r) + \left[ \frac{2\mu}{h^2} (E - \frac{1}{2} k_\beta r^{\beta - 1}) - \frac{l(l + \alpha - 2)}{r^2} \right] R(r) = 0, \quad (2.2) \]
where \( k_\beta |_{\beta=3} = \omega^2 \mu \) and the convergent solution should be for \( (r \in (0, \infty)) \).

By making the second change of variable, namely
\[ R(r) = r^l \left[ e^{-\frac{\mu \omega}{2h} r^2} \right] y(r), \]
we obtain the differential equation corresponding to \( y(r) \) as follows
\[ y''(r) + \frac{1}{r} [(2l + \alpha - 1) - \frac{2\mu \omega^2 r}{h}] y'(r) \]
\[ + \left[ \left( -\frac{\mu \omega}{h} (2l + \alpha) + \frac{2\mu}{h^2} (E - \frac{1}{2} k_\beta r^{\beta - 1}) + \frac{\mu^2}{h^2} \omega^2 r^2 \right) y(r) = 0 \right], \quad (2.3) \]

Finally, by using the new variables \( x = \frac{\mu \omega^2}{h} \) and \( k^2 = \frac{2\mu E}{h^2} \) the equation (2.3) becomes
\[ x \frac{d^2 y}{dx^2} + (A - x) \frac{dy}{dx} + \left( \frac{B}{2} - \frac{x}{4} - \frac{C^2}{2} \right) y(x) = 0, \quad (2.4) \]
where \( A = \frac{2l + \alpha}{2}, \quad B = \frac{k^2 h^2}{4 \mu \omega} \) and \( C = \frac{k_\beta}{h \omega} \left( \frac{h}{\mu \omega} \right)^{\frac{\beta - 1}{2}} \).

3. EXACT SOLUTIONS OF THE RADIAL DIFFERENTIAL EQUATION

Let us take \( y(x) = e^{x/2} F(x) \). Substituting \( y(x), y'(x) \) and \( y''(x) \) into the differential equation (2.4), we may obtain
\[ x \frac{d^2 F}{dx^2} + A \frac{dF}{dx} + \left( \frac{B}{2} - \frac{C}{4} x^{\frac{\beta - 1}{2}} \right) F(x) = 0. \quad (3.5) \]

Our aim is to solve (3.5). The first step is to find the regular singular points for equation (3.5).

Lemma. \( x_0 = 0 \) is a regular singular point for equation (3.5) if and only if \( \beta \geq -1 \).

Proof. Since \( \alpha_0 = \lim_{x \to 0} x A/x = A \) and \( \beta_0 = \lim_{x \to 0} x^{\beta+1} B - C \frac{x^{\beta+1}}{4x^{\frac{\beta+1}{2}}} = -C/4 \lim_{x \to 0} x^{\frac{\beta+1}{2}} \)
are both finite if and only if \( \beta \geq -1 \). Thus, \( x_0 = 0 \) is a regular singular point if and only if \( \beta \geq -1 \).

Case 1. For \( \beta = -1 \), the Equation (3.5) becomes
\[ x^2 \frac{d^2 F}{dx^2} + Ax \frac{dF}{dx} + \left( Bx - \frac{C}{4} \right) F(x) = 0. \quad (3.6) \]

The next step is to use the substitution \( F(x) = x^{\frac{1-\alpha}{2}} G(x) \) which leads to the
following equation
\[ x^2 G''(x) + x G'(x) + \left( Bx - \frac{A^2 - 2A + 1 + C}{4} \right) G(x) = 0. \]  
(3.7)

This equation can be written in the form
\[ x^2 G''(x) + x G'(x) + \frac{1}{4} \left[ \left( 2\sqrt{B} \sqrt{x} \right)^2 - \left( \sqrt{(A-1)^2 + C} \right)^2 \right] G(x) = 0, \]
which is the well-known Bessel’s equation of the first kind and it has the general solution
\[ G(x) = C_1 J_\nu \left( 2\sqrt{B} \sqrt{x} \right) + C_2 Y_\nu \left( 2\sqrt{B} \sqrt{x} \right), \]
where \( \nu = \sqrt{(A-1)^2 + C} \) and
\[ J_\nu(t) = \sum_{m=0}^\infty \frac{(-1)^m 2m+\nu}{m! \Gamma(\nu + m + 1)}, \quad Y_\nu(t) = \frac{J_\nu(t) \cos \nu \pi - J_{-\nu}(t)}{\sin \nu \pi}. \]

Therefore, corresponding to \( \beta = -1 \), equation (2.4) has a general solution of the form
\[ y(x) = e^{x/2} x^{1/4} \left\{ C_1 J_\nu \left( 2\sqrt{B} x \right) + C_2 Y_\nu \left( 2\sqrt{B} x \right) \right\}. \]

Case 2. For \( \beta = -\frac{1}{2} \), the equation (3.5) becomes
\[ x^2 \frac{d^2 F}{dx^2} + A x \frac{dF}{dx} + \left( B x - \frac{C}{4} x^{1/4} \right) F(x) = 0. \]  
(3.8)

For this equation, we use the following substitution \( t = 2 \sqrt{-Bx} \).

After doing some calculations, we arrive at the differential equation
\[ t^2 \frac{d^2 F}{dt^2} + (2At - t) \frac{dF}{dt} - \left( t^2 + \frac{C}{(4B)^{1/4}} \sqrt{t} \right) F(t) = 0 \]  
(3.9)

By using the transformation \( F(t) = e^t G(t) \), the above differential equation may take the new form
\[ t^2 \frac{d^2 G}{dt^2} + \left( 2t^2 + 2At - t \right) \frac{dG}{dt} + \left( 2At - t - \frac{C}{(4B)^{1/4}} \sqrt{t} \right) G(t) = 0. \]

The next step is to use the following new variable \( w = \sqrt{t} \). Therefore, the obtained differential equation can be written as
\[ \frac{d^2 G}{dw^2} + \frac{4w^2 + 4A - 3}{w} \frac{dG}{dw} + \frac{8Aw - 4w - 4}{w} \frac{C}{(4B)^{1/4}} G(w) = 0. \]
Finally we use the linear transformation \( w = \frac{i}{\sqrt{2}} z \) where \( i = \sqrt{-1} \).

Now, in terms of the new variable \( z \), the previous differential equation can be written as

\[
\frac{d^2 G}{dz^2} - \frac{2z^2 - 1 + 4 - 4A}{z} \frac{dG}{dz} - \frac{1}{2} \left( \frac{8A - 4}{z} \right) + \frac{C}{(-B)^{1/4}} G(z) = 0. \tag{3.10}
\]

This is the well-known HeunB equation with \( \alpha = 4A - 4, \beta = 0, \gamma = 0 \) and \( \delta = \frac{4C}{(-B)^{1/4}} \). Hence, corresponding to \( \beta = -\frac{1}{2} \), the general solution of equation (2.4) is the expression

\[
y(x) = e^{(x + 4\sqrt{-Bx})/2} \left\{ C_1 \text{HeunB} \left( 4A - 4, 0, 0, -\frac{4iC}{(-B)^{1/4}}; 2i(-Bx)^{1/4} \right) + C_2 \text{HeunB} \left( 4 - 4A, 0, 0, -\frac{4iC}{(-B)^{1/4}}; 2i(-Bx)^{1/4} \right) x^{1-A} \right\}.
\]

**Case 3.** For \( \beta = 0 \) the equation (3.5) becomes the differential equation;

\[
x \frac{d^2 F}{dx^2} + A \frac{dF}{dx} + \left( B - \frac{C}{4} x^{-1/2} \right) F(x) = 0. \tag{3.11}
\]

For this equation, we use the same substitution which is used in case 1, that is, \( t = -2i\sqrt{Bx} \), where \( i = \sqrt{-1} \).

After substituting these expressions into equation (2.5) and doing some simplifications, one may arrive at the differential equation

\[
t \frac{d^2 F}{dt^2} + (2A - 1) \frac{dF}{dt} - \left( t + \frac{iC}{2\sqrt{B}} \right) F(t) = 0.
\]

By using the transformation \( F(t) = e^t G(t) \), the above differential equation may take the new form

\[
t \frac{d^2 G}{dt^2} + \left( t + \frac{2A - 1}{2} \right) \frac{dG}{dt} - \left( \frac{1}{2} - \frac{2A}{2} + \frac{iC}{4\sqrt{B}} \right) G(t) = 0. \tag{3.12}
\]

By using the following change of variable \( z = -2t \), one may write the equation (3.12) reduced to the following form

\[
z \frac{d^2 G}{dz^2} + (-z + 2A - 1) \frac{dG}{dz} - \left( \frac{1}{2} - \frac{2A}{2} + \frac{iC}{4\sqrt{B}} \right) G(z) = 0. \tag{3.13}
\]

This is the well-known Hypergeometric equation in the general form. Thus, the
general solution of (2.4) corresponding to $\beta = 0$, is the expression
\[
y(x) = e^{(x-4i\sqrt{B}x)/2} \left\{ C_1 \text{Hypergeom} \left( \left[ \frac{2A-1}{2} - i\frac{C}{4\sqrt{B}} \right], [2A-1], 4i\sqrt{B}x \right) + C_2 \text{Hypergeom} \left( \left[ -\frac{2A}{2} - i\frac{C}{4\sqrt{B}} \right], [3-2A], 4i\sqrt{B}x \right) x^{1-A} \right\}.
\]

**Case 4.** In the following we consider the case $\beta = 1$. Therefore, equation (3.5) becomes the differential equation
\[
x^2 \frac{d^2 F}{dx^2} + A x \frac{dF}{dx} + \left( B x - \frac{C}{4} \right) F(x) = 0. \quad (3.14)
\]
If we use the substitution $F(x) = x^{1-A} G(x)$ and after doing some simplifications, we arrive at the differential equation
\[
x^2 G''(x) + x G'(x) + \frac{1}{4} \left( \left( \sqrt{4B-C} \sqrt{x} \right)^2 - (A-1)^2 \right) G(x) = 0, \quad (3.15)
\]
which is the well-known Bessel’s equation of the first kind and it has the general solution as given below
\[
G(x) = C_1 J_\nu \left( \sqrt{4B-C} \sqrt{x} \right) + C_2 Y_\nu \left( \sqrt{4B-C} \sqrt{x} \right),
\]
where $\nu = A - 1$, 
\[
J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+\nu}}{2^{2m+\nu} m! \Gamma(\nu + m + 1)}, \quad Y_\nu(t) = \frac{J_\nu(t) \cos \nu \pi - J_{-\nu}(t)}{\sin \nu \pi}.
\]
Therefore, corresponding to $\beta = 1$, equation (2.4) has a general solution of the form
\[
y(x) = e^x/2 x^{1-A} \left\{ C_1 J_\nu \left( \sqrt{4B-C} \sqrt{x} \right) + C_2 Y_\nu \left( \sqrt{4B-C} \sqrt{x} \right) \right\}.
\]

**Case 5.** For $\beta = 3$, the equation (3.5) becomes the differential equation
\[
x x^2 \frac{d^2 F}{dx^2} + A \frac{dF}{dx} + \left( B - \frac{C}{4} \right) F(x) = 0. \quad (3.16)
\]
For this equation, we use the substitution $F(x) = x^{-A/2} G(x)$.

After substituting $F(x)$, $F'(x)$ and $F''(x)$ into equation (3.16) and doing some simplifications, we arrive at the differential equation
\[
x^2 \frac{d^2 G}{dx^2} + \left( \frac{2A-A^2}{4} + Bx - \frac{C}{4 x^2} \right) G(x) = 0.
\]
By using the linear transformation $t = x\sqrt{C}$, the above differential equation may
take the new form
\[ G''(t) + \left( -\frac{1}{4} + \frac{B}{t} + \frac{1}{t^2} \left[ \frac{A-1}{4} \right]^2 \right) G(t) = 0. \quad (3.17) \]

Since the well-known Whittaker’s equation
\[ G''(z) + \left( -\frac{1}{4} + a + \frac{1-b^2}{z^2} \right) G(z) = 0 \quad (3.18) \]
has general solution
\[ G(z) = C_1 \text{WhittakerM} \left( \frac{B}{\sqrt{C}}, \frac{A-1}{2}, \sqrt{C} x \right) + C_2 \text{WhittakerW} \left( \frac{B}{\sqrt{C}}, \frac{A-1}{2}, \sqrt{C} x \right). \quad (3.19) \]
it implies that equation (2.8) has general solution
\[ G(x) = C_1 \text{WhittakerM} \left( \frac{B}{\sqrt{C}}, \frac{A-1}{2}, \sqrt{C} x \right) + C_2 \text{WhittakerW} \left( \frac{B}{\sqrt{C}}, \frac{A-1}{2}, \sqrt{C} x \right). \quad (3.20) \]

Therefore, the solution of equation (3.17) becomes
\[ F(x) = x^{-\frac{3}{2}} \left( C_1 \text{WhittakerM} \left( \frac{B}{\sqrt{C}}, \frac{A-1}{2}, \sqrt{C} x \right) + C_2 \text{WhittakerW} \left( \frac{B}{\sqrt{C}}, \frac{A-1}{2}, \sqrt{C} x \right) \right). \]

Hence, corresponding to \( \beta = 3 \), the general solution of equation (2.4) is the expression
\[ y(x) = e^{x/2} x^{-\frac{3}{2}} \left( C_1 \text{WhittakerM} \left( \frac{B}{\sqrt{C}}, \frac{A-1}{2}, \sqrt{C} x \right) + C_2 \text{WhittakerW} \left( \frac{B}{\sqrt{C}}, \frac{A-1}{2}, \sqrt{C} x \right) \right). \]

**Case 6.** For \( \beta = 7 \), the equation (2.4) becomes
\[ x \frac{d^2 F}{dx^2} + A \frac{dF}{dx} + \left( B - \frac{C}{4} x^3 \right) F(x) = 0. \quad (3.21) \]

Using the substitution \( t = \frac{\sqrt{C}}{4} x^2 \) we obtain the differential equation
\[ 2t^2 \frac{d^2 F}{dt^2} + (At + t) \frac{dF}{dt} - \left( 2t^2 - \frac{B}{C^{1/4}} \sqrt{t} \right) F(t) = 0. \quad (3.22) \]

Now, the simple transformation \( F(t) = e^t G(t) \), reduces the above differential equation to the form
\[ 2t^2 \frac{d^2 G}{dt^2} + (At^2 + At + t) \frac{dG}{dt} + \left( At + t + \frac{B}{C^{1/4}} \sqrt{t} \right) G(t) = 0. \quad (3.23) \]
Taking $w = \sqrt{t}$ and after using some simplifications, the last differential equation can be written in the form

$$w \frac{d^2 G}{dw^2} + \left(4w^2 + A \right) \frac{dG}{dw} + \left(2w + 2Aw + 2 \frac{B}{C^{1/4}} \right) G(w) = 0. \tag{3.24}$$

Finally, the linear transformation $w = \frac{i}{\sqrt{2}} z$, where $i = \sqrt{-1}$ gives, in terms of the new variable $z$, the following differential equation

$$\frac{d^2 G}{dz^2} - \frac{2z^2 - 1 + A}{z} \frac{dG}{dz} - \frac{1}{2} \frac{(2A + 2) z - 2\sqrt{2Bi} C^{1/4}}{z} G(z) = 0. \tag{3.25}$$

This is the well-known HeunB equation with $\alpha = A - 1$, $\beta = 0$, $\gamma = 0$ and $\delta = -\frac{2B\sqrt{2i}}{C^{1/4}}$. Hence, corresponding to $\beta = 7$, the general solution of equation (2.4) is the expression

$$y(x) = e^{(2x + \sqrt{C}x^2)/4} \left\{ C_1 \text{HeunB} \left( A - 1, 0, 0, \frac{2\sqrt{2Bi}}{C^{1/4}}, \frac{\sqrt{2i}}{2}, C^{1/4}, x \right) ight. \\
+ C_2 \text{HeunB} \left( 1 - A, 0, 0, \frac{2\sqrt{2Bi}}{C^{1/4}}, -\frac{\sqrt{2i}}{2}, C^{1/4}, x \right) \left. \right\} x^{1-A}.$$

4. CONCLUSIONS

Fractional dynamics started to play an important role in several branches of science and engineering. In this paper, by using suitable change of variables, we have obtained the exact solutions of the Schrödinger equation corresponding to the fractional dimensional harmonic oscillator.

It was proved that $x_0 = 0$ is a regular singular point for equation (3.5) if and only if $\beta \geq -1$.

An interesting point to be specified here is that for the special case $\alpha = 1$, $\beta \leq 1$ we obtained the case which is called spiked harmonic oscillator.

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