Fullerenes are molecules in the form of cage-like polyhedra, consisting solely of carbon atoms. In this paper, the Omega and Sadhana polynomials of a new infinite class of fullerenes constructed by leapfrog principle is computed.

Key words: Omega Polynomial, Sadhana Polynomial, Fullerene Graph.

1. INTRODUCTION

A fullerene graph is a cubic 3-connected plane graph with (exactly 12) pentagonal faces and hexagonal faces. Let $F_n$ be a fullerene graph with $n$ vertices. By the Euler formula one can see that $F_n$ has 12 pentagonal and $n/2 - 10$ hexagonal faces.

Let $G = (V, E)$ be a connected graph with the vertices set $V = V(G)$ and the edges set $E = E(G)$, without loops and multiple edges. The distance $d(x,y)$ between $x$ and $y$ is defined as the length of a minimum path between $x$ and $y$. Two edges $e = ab$ and $f = xy$ of $G$ are called codistant, "$e$ co $f$", if and only if $d(a,x) = d(b,y) = k$ and $d(a,y) = d(b,x) = k + 1$ or vice versa, for a non-negative integer $k$. It is easy to see that the relation "co" is reflexive and symmetric but it is not necessary to be transitive. Set $C(e) = \{f \in E(G) : f \ co \ e\}$. If the relation "co" is transitive on $C(e)$ then $C(e)$ is called an orthogonal cut "oc" of the graph $G$. The graph $G$ is called co-graph if and only if the edge set $E(G)$ a union of disjoint orthogonal cuts. If any two consecutive edges of an edge-cut sequence are topologically parallel within the same face of the covering, such a sequence is called a quasi-orthogonal cut qoc strip. The Omega polynomial has been defined on the ground of $qoc$ strips [1-5]:

$$\Omega(G, x) = \sum m_x e^x.$$
In a counting polynomial, the first derivative (in $x = 1$) defines the type of property which is counted, namely:

$$\Omega(G,1) = \sum c.m.c = e = |E(G)|.$$ 

The Sadhana index $Sd(G)$ for counting $qoc$ strips in $G$ was defined by Khadikar et al. [6-9] as $Sd(G) = \sum c.m(G,c)(|E|-c)$. We now define the Sadhana polynomial of a graph $G$ as $Sd(G,x) = \sum c.mx^{E-c}$. By definition of Omega polynomial, one can obtain the Sadhana polynomial by replacing $x^c$ with $x^{(|E|-c)}$ in omega polynomial. Then the Sadhana index will be the first derivative of $Sd(G,x)$ evaluated at $x = 1$.

Let $G$ be a fullerene graph on $n$ vertices. A leapfrog transform $G'$ of $G$ is a graph on $3n$ vertices obtained by truncating the dual of $G$. Hence, $G' = Tr(G^*)$, where $G^*$ denotes the dual of $G$. It is easy to check that $G'$ itself is a fullerene graph. We say that $G'$ is a leapfrog fullerene obtained from $G$ and write $G' = Le(G)$. In the other word, for a given fullerene $F_n$ put an extra vertex into the centre of each face of $F_n$. Then connect these new vertices with all the vertices surrounding the corresponding face. Then the dual polyhedron is again a fullerene having $3n$ vertices 12 pentagonal and $(3n/2)-10$ hexagonal faces. From Fig. 1, one can see that $Le(C_{20}) = C_{60}$. For a more thorough introduction and treatment of leapfrog fullerenes we refer the reader to [10 - 12]. Through this paper all notations are standard and mainly taken from [13, 14].

![Fig. 1 – The leapfrog of graphs $F_{24}$ and $F_{30}$](image-url)
2. MAIN RESULTS AND DISCUSSIONS

The aim of this paper is computing Omega and Sadhana polynomials of leapfrog fullerene $F_{24,3^n}$ constructed by $F_{24}$. In the other word by using the leapfrog principle we can construct an infinite class of fullerenes and so we compute the Omega and Sadhana polynomials of $F_{24,3^n}$. To do it at first we should consider the following examples.

**Example 1.** Let $F_{20}$ be a fullerene with 20 vertices depicted in Fig. 2. It is easy to see that $|E(F_{20})| = 30$. By computing the quasi-orthogonal cut qoc strips of $F_{20}$ one can see that the Omega and Sadhana polynomials are as $Ω(F_{20}, x) = 30x$ and $Sd(F_{20}, x) = 30x^{39}$.

![Fig. 2 – The graph of fullerene $F_{20}$](image)

**Example 2.** Consider the fullerene graph $F_{24}$. This fullerene graph has 36 edges. Similar to example 1 one can see that $Ω(F_{24}, x) = 24x + 6x^3$ and so, $Sd(F_{24}, x) = 24x^{35} + 6x^{34}$. In Fig. 3 one can see the $F_{24}$ and $Le(F_{24})$.

![Fig. 3 – The leapfrog of graph $F_{24}$](image)

By continuing this method we achieve the graph of fullerene $F_{24,3^n}$. For computing the Omega and Sadhana polynomials we have to consider two cases. At first let $n$
be an even number. By Fig. 4(ii), it is easy to see that there are four types of edges for qoc strips. We denote them by $e_1$, $e_2$, $e_3$ and $e_4$ in which $|C(e_1)| = 3^{n/2}$, $|C(e_2)| = 2 \times 3^{n/2}$, $|C(e_3)| = 2 \times 3^{n/2+1}$ and $|C(e_4)| = 10 \times 3^{n/2+1}$. In the other word there are 24, 6, 3$^{n/2-1}$ and 3$^{n/2-1}$ edges of type $e_1$, $e_2$, $e_3$ and $e_4$, respectively. Now let $n$ be an odd number Fig. 4(i). By the same way we can see there are three types of edges for qoc strips. We name them by $e_1$, $e_2$, and $e_3$. It is not difficult to see $|C(e_1)| = 3^{(n+1)/2}$, $|C(e_2)| = 2 \times 3^{(n+1)/2}$ and $|C(e_3)| = 2 \times 3^{(n+5)/2}$ and there are 24, 6, 3$^{n/2-1}$ and 2$\times 3^{(n-1)/2-1}$ edges of type $e_1$, $e_2$, and $e_3$, respectively. Hence, we proved the following theorem:

**Theorem.** Consider the fullerene graph $F_{24x3^n}$ $(n \geq 3)$ depicted in Fig. 4. Then the Omega polynomial is as follows:

$$\Omega(F_{24x3^n}, x) = \begin{cases} 
24x^{3^{n/2}} + 6x^{2\times3^{n/2}} + (3^{n/2} - 1)(x^{2\times3^{(n+1)/2}} + x^{10\times3^{(n+1)/2}}) 2 \mid n \\
24x^{3^{(n+1)/2}} + 6x^{2\times3^{(n+1)/2}} + 2(3^{(n-1)/2} - 1)x^{2\times3^{(n+5)/2}} 2 \mid n 
\end{cases}$$

**Corollary.** For the fullerene graph $F_{24x3^n}$ $(n \geq 3)$ the Sadhana polynomial is as follows:

$$Sd(F_{24x3^n}, x) = \begin{cases} 
24x^{E(3^{n/2})} + 6x^{E(2\times3^{n/2})} + (3^{n/2} - 1)(x^{E(2\times3^{(n+1)/2})} + x^{E(10\times3^{(n+1)/2})}) 2 \mid n \\
24x^{E(3^{(n+1)/2})} + 6x^{E(2\times3^{(n+1)/2})} + 2(3^{(n-1)/2} - 1)x^{E(2\times3^{(n+5)/2})} 2 \mid n 
\end{cases}$$

Fig. 4 – (i). The graph of $F_{24x3^n}$ for $n = 3$. 
Fig. 4 – (ii). The graph of $F_{24,5}$ for $n = 4$. 
REFERENCES