ON COSMOLOGIES WITH NON-MINIMALLY COUPLED SCALAR FIELDS, 
THE “REVERSE ENGINEERING METHOD” AND THE EINSTEIN FRAME

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This article is mainly dedicated to the so-called "Reverse Engineering Method" (reconstructing the shape of the potential in cosmologies based on a scalar field and starting with the time behavior of the scale factor) recently developed by Ellis and Madsen [1] and other authors [2, 3], this time for a cosmology with a scalar field non-minimally coupled with gravity. It is more convenient to perform a conformal transformation of the metric to the so called Einstein frame [4]. In Einstein frame the new redefined scalar field appears to be only minimally coupled, thus we can proceed with REM as was prescribed by Ellis and Madsen and then transform back the main operators in the normal frame. We processed several examples pointing out the influence of the non-minimal coupling on the evolution of the universe.

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1. INTRODUCTION

In this article we consider the so-called "Reverse Engineering Method" (reconstructing the shape of the potential in cosmologies based on a scalar field and starting with the time behavior of the scale factor) developed by Ellis and Madsen [1], and other authors [2, 3]. This time we deal with a cosmology with a scalar field non-minimally coupled with gravity. Recently [3] we processed the REM for the scalar field non-minimally coupled with gravity in a direct manner, as we obtained the new Friedmann equations for these types of cosmologies. As the REM method cannot be easily applied for this new environment we graphically processed several type of cosmologies (with exponential and linear expansion) proving the possibility of applying REM. Here we followed a different method. The starting point is a conformal transformation to the so called Einstein frame [4] were, after a proper redefining of the scalar fields, it turns out that the theory is now based on a minimally coupled scalar field, in fact the new one! Thus within the Einstein frame
we can apply the REM using the results summarized in [2] – mainly in the Table 1 in the mentioned article. Of course we then need to return back the main values and operators to the original frame which will give us the potential in terms of the scalar field.

The article is organized as follows: in the next section we summarize the method of introducing the Einstein frame as was described in detail in [4] and in the articles cited there. We output the main transformation relations we need to deal with the theory in the Einstein frame. The section 3 contains the main results, namely the two examples we graphically processed: the exponential expansion universe and the ekpyrotic universe. The article ends with a short section containing the main conclusions.

2. THE EINSTEIN FRAME AND THE REVERSE ENGINEERING METHOD

We start here with a model where an inflaton field $f$ is non-minimally coupled with a scalar curvature $R$ [4]

$$L = \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) - \frac{1}{2} \xi R \phi^2 \right],$$

(1)

where $G$ is the Newton's gravitational constant, and $\xi$ is a coupling constant, $R$ being the Ricci scalar and $g$ the determinant of the metric tensor.

Although we can proceed with the reverse method directly with the Friedmann eqs. obtained from this Lagrangian (as it was done in [3] it is rather complicated due to the existence of non-minimal coupling). In [3] we appealed to the numerical and graphical facilities of Maple platform. It is more convenient to transform to the Einstein frame by performing a conformal transformation

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu},$$

(2)

where

$$\Omega^2 = 1 - \xi 8\pi \phi^2.$$  

(3)

Then we obtain the following equivalent Lagrangian:

$$L = \sqrt{-\hat{g}} \left[ \frac{1}{16\pi G} \hat{R} - \frac{1}{2} \hat{F}^2 \left( \dot{\Phi} \right)^2 - \dot{V}(\phi) \right],$$

(4)

where variables with a caret denote those in the Einstein frame,

$$F^2 = \frac{1 - (1 - 6\xi 8\pi \phi^2)^2}{1 - 8\pi \xi \phi^2}, \quad \dot{V}(\phi) = \frac{V(\phi)}{(1 - 8\pi \xi \phi^2)^2}. $$

(5)

Introducing a new scalar field $\Phi$ as
\[ \Phi = \int F(\dot{\phi})d\phi, \quad (6) \]

the Lagrangian in the new frame is reduced to the canonical form:

\[ L = \sqrt{-g} \left[ \frac{1}{16\pi G} \dot{R} - \frac{1}{2} \left( \dot{\Phi} \right)^2 - \ddot{\Phi} \right]. \quad (7) \]

Thus we arrive to the main conclusion: we can process a REM in the Einstein frame (using the results from the minimally coupling case and then we can convert the results in the original frame).

Before going forward with some concrete results, let’s investigate some important equations for processing the transfer from Einstein frame to the original one. First the main coordinates are, using as usually the Friedmann-Robertson-Walker metric:

\[ \dot{t} = \int \Omega dt \quad \text{and} \quad \dot{R}(t) = \Omega R(t), \quad (8) \]

where \( \dot{R}(t) \) is the scale factor of the universe. Then the new scalar field \( \Phi \) can be obtained by integrating its above expression, namely

\[ \Phi = \frac{\sqrt{3}}{2\sqrt{\pi}} \text{sgn}(\xi) \text{tanh}^{-1} \left[ \frac{4\sqrt{3}\pi \phi \xi \text{sgn}(\xi)}{\sqrt{1 - (1 - 6\xi)^2} 8\pi \phi^2} \right] + \frac{\sqrt{2}}{4\sqrt{\pi \xi}} \left[ \sqrt{\xi(1 - 6\xi)} \sin^{-1}(2\phi\sqrt{2\pi \xi(1 - 6\xi)}) \right], \quad (9) \]

where \( \text{sgn}(\xi) \) represents the sign of \( \xi \)—namely +1 or −1.

3. GRAPHICAL RESULTS IN CERTAIN CASES

For the next graphical results we processed certain examples inside the Einstein frame using the Table 1 from [2] where we used \( \Phi \) instead of \( \phi \) and all the values with caret instead of \( t, V, \) etc. Then we investigated the graphical shape of the potential in the original frame in terms of the scalar field \( \phi \) using the transformations described in the previous section.

3.1. THE EXPONENTIAL EXPANSION

This is case no. 1 from the Table. 1 of [2], namely the scale factor in Einstein frame depends exponentially of the time in the Einstein frame:
Fig. 1 – The potential in terms of the scalar field for $\omega = 1$, $\xi = 0$ (with green line in both panels) and for $\xi = -0.1$ (left panel) and $\xi = 0.1$ (right panel) with blue line.

Fig. 2 – The potential in terms of the scalar field and $\omega$, for $\xi = 0$ (the green surface in both panels) and for $\xi = -0.1$ (left panel) and $\xi = 0.1$ (right panel) the blue surfaces

3.2. THE EKPYROTIC UNIVERSE

This is example no. 6 from Table 1 of [2] having:

$$\tilde{R}(\tilde{t}) = R_0 \exp(\omega \tilde{t})$$

and

$$V(\Phi) = 2 \left[ B \cosh \left( \frac{\omega \Phi}{B} \right) \right]^2 - \frac{3\omega^2}{4\pi},$$

where
\[ B^2 = \frac{1}{4\pi} \left( 1 + \frac{k}{R_0^2} \right). \]

Then, using the transformations relations from the previous sections we obtained the graphical results outpointed in the next figures 3 and 4.

**Fig. 3** – The ekpyrotic universe: the potential in terms of the scalar field for \( \omega = 1 \) and \( k = 1 \), for \( \xi = 0 \) (the green lines in both panels) and for \( \xi = -0.1 \) (left panel) and \( \xi = 0.1 \) (right panel) the blue lines.

**Fig. 4** – The ekpyrotic universe: the potential in terms of the scalar field and \( \omega \), for \( k = 1 \) and \( \xi = 0.05 \).

**4. OUTLINE AND MAIN CONCLUSIONS**

This article is a continuation of our investigations on the applicability of REM for cosmologies with scalar field non-minimally coupled with gravity. Due to the much more complicated Friedmann equations we were faced in this case, we choose, this time, to study the REM in the so-called Einstein frame – an idea introduced some time ago in the study of non-minimally coupled scalar fields [4].
In the Einstein frame – obtained after a conformal transformation applied to the theory – the action of the system becomes (for the new redefined scalar field) the same as for a minimally coupled scalar field. Thus we applied the REM inside the Einstein frame as we done in the original frame and we described in detail in [2]. Actually we processed some of the examples in Table 1 from [2] and then transformed the potential in terms of the scalar field back to the original frame using the transformation relations given above and in [4]. Then we graphically processed the results as we done in [3] pointing out the difference between the potential with or without non-minimal coupling. The main conclusion is that the coupling factor (which describes the intensity of the coupling between the scalar field and gravity) is of a great importance. It is very obvious the non-minimal coupling influences drastically the shape of the potential and thus the character of the theory governing the universe with such a scalar field.

Application of this approach to the tachyonic potentials [2] and investigation of their quantum behavior [5] will be published elsewhere.

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