SHOCK WAVE SOLUTION OF BENNEY-LUKE EQUATION

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This paper obtains the shock wave solution, also known as topological 1-soliton solution, of the Benney-Luke equation. The ansatz method is going to be used to integrate the equation for the solution. The parameter restrictions will also fall out naturally in course of derivation of the solution.

Key words: the Benney-Luke equation, the ansatz method, topological 1-soliton solution.

1. INTRODUCTION

The nonlinear evolution equations (NLEEs) that are studied in Theoretical Physics, especially in the context of wave phenomena, leads to various forms of wave solutions. They are solitary waves, shock waves, cnoidal waves, snoidal waves, cuspons, peakons, stumpons and various other types. These waves appear in various scenarios in daily real life situations. For example, solitons appear in the propagation of pulses through optical fibers while shock waves appear in the supersonic jet flow. Another example is where cnoidal waves appear in shallow water waves although an extremely rare phenomena.

The NLEEs, that appear in the context of wave dynamics, have various methods of integrating them. Besides the classical and most powerful method, namely the Inverse Scattering Transform (IST), there are various methods that have been developed in the last couple of decades, which integrates these NLEEs and...
wave solutions can be extracted. Some of these techniques are variational iteration method, Adomian decomposition method, homotopy analysis method, exponential function method, traveling wave solutions, semi-inverse variational method. In this paper, there will be one such method, namely the ansatz method will be employed to extract the shock wave solution of a NLEE.

Shock wave is a disturbance that propagates through a media. It actually represents a sharp discontinuity of the parameters that describe the media, namely the pressure, temperature and density. Unlike solitons, where the energy is a conserved quantity and thus stays constant during its propagation, shock waves dissipate energy relatively quickly with distance. One source of a shock wave is when the supersonic jets fly at a speed that is greater than the speed of sound. This results in the drag force on aircraft with shocks. It needs to be noted that shock wave solution corresponds to a discontinuous weak solution associated with a hyperbolic system.

2. ANSATZ METHOD

The NLEE that is going to be studied in this paper is called Benney-Luke (BL) equation and is given by [1-10]

\[ u_{tt} - k^2 u_{ss} + au_{xxxx} + bu_{xxt} + cu_{ss} + du_{xt} = 0 \]  

(1)

BL equation is a NLEE that has been around and studied for a very long time. There are various analysis that was conducted for this equation. These are the stability analysis [5, 8], Cauchy problem [1, 7, 10], existence and analyticity of solutions [4], traveling wave solutions [2], the generalized two-dimensional BL equation [6] and so forth. In this paper a topological 1-soliton solution to (1), also known as shock wave solution will be obtained. The starting hypothesis for 1-soliton solution is given by

\[ u(x,t) = A \tanh^{p} \tau \]  

(2)

where

\[ \tau = B(x - vt) \]  

(3)

Here in (2) and (3), \( A \) and \( B \) are free parameters while \( v \) is the velocity of propagation of the wave. Also, the unknown exponent \( p \) will be determined during the course of the derivation of the shock wave solution to (1). It needs to be noted that typically equation (2) models the mathematical structure of a topological soliton in various physical situations [2]. Therefore, from (1),

\[ u_{g} = pAv^{2}B^{2} \{(p - 1)\tanh^{p-2}\tau - 2p\tanh^{p}\tau + (p + 1)\tanh^{p+2}\tau\} \]  

(4)
\[ u_{st} = pAB^2 \left\{ (p - 1) \tanh^{p-2} \tau - 2p \tanh^p \tau + (p + 1) \tanh^{p+2} \tau \right\} \]  (5)  
\[ u_{stt} = pAB^2 \left[ (p - 1)(p - 2)(p - 3) \tanh^{p-4} \tau + (p + 1)(p + 2)(p + 3) \tanh^{p+4} \tau \right. 
\left. - 2 \left\{ p^2 + (p - 2)^2 \right\} (p - 1) \tanh^{p-2} \tau - 2 \left\{ p^2 + (p + 2)^2 \right\} (p + 1) \tanh^{p+2} \tau + 
+ \left\{ 4p^3 + (p - 1)^2(p - 2) + (p + 1)^2(p + 2) \right\} \tanh^p \tau \right] \]  (6)  
\[ u_{xst} = pAB^2 v^2 \left[ (p - 1)(p - 2)(p - 3) \tanh^{p-4} \tau + (p + 1)(p + 2)(p + 3) \tanh^{p+4} \tau \right. 
\left. - 2 \left\{ p^2 + (p - 2)^2 \right\} (p - 1) \tanh^{p-2} \tau - 2 \left\{ p^2 + (p + 2)^2 \right\} (p + 1) \tanh^{p+2} \tau + 
+ \left\{ 4p^3 + (p - 1)^2(p - 2) + (p + 1)^2(p + 2) \right\} \tanh^p \tau \right] \]  (7)  
\[ u_{st} = p^2 A^2 B^3 v \left\{ (p + 1) \tanh^{2p+2} \tau - (p - 1) \tanh^{2p-2} \tau + 
+ (3p - 1) \tanh^{2p-2} \tau - (3p + 1) \tanh^{2p+2} \tau \right\} \]  (8)  
\[ u_{st} = p^2 A^2 B^3 v \left\{ (p + 1) \tanh^{2p+2} \tau - (p - 1) \tanh^{2p-2} \tau + 
+ (3p - 1) \tanh^{2p-2} \tau - (3p + 1) \tanh^{2p+2} \tau \right\} \]  (9)  

Substituting (4)-(9) into (1) gives
\[ pA \left( v^2 - k^2 \right) B^2 \left\{ (p - 1) \tanh^{p-2} \tau - 2p \tanh^p \tau + (p + 1) \tanh^{p+2} \tau \right\} + 
\left\{ (a + bv^2) pAB^4 (p - 1)(p - 2)(p - 3) \tanh^{p-4} \tau + 
+(p + 1)(p + 2)(p + 3) \tanh^{p+4} \tau \right. 
\left. - 2 \left\{ p^2 + (p - 2)^2 \right\} (p - 1) \tanh^{p-2} \tau - 2 \left\{ p^2 + (p + 2)^2 \right\} (p + 1) \tanh^{p+2} \tau + 
+ \left\{ 4p^3 + (p - 1)^2(p - 2) + (p + 1)^2(p + 2) \right\} \tanh^p \tau \right] + 
\left\{ (c + d) p^2 A^2 B^3 v \left\{ (p + 1) \tanh^{2p+2} \tau - (p - 1) \tanh^{2p-2} \tau + 
+ (3p - 1) \tanh^{2p-2} \tau - (3p + 1) \tanh^{2p+2} \tau \right\} \right\} = 0 \]  (10)  

From (10), equating the exponents \( 2p + 3 \) and \( p + 4 \) gives \( 2p + 3 = p + 4 \)  (11)
so that \( p = 1 \)  (12)  
Now, the same value of \( p \) is obtained when the following pair of exponents \( 2p + 1 \) and \( p + 2 \), \( 2p - 1 \) and \( p \), \( 2p - 3 \) and \( p - 2 \) are equated with each other.
Finally, from (10), the linearly independent functions are \( \tanh^{p+j}\tau \) for \( j = \pm 2, 0, \pm 4 \). Hence, setting their respective coefficients to zero yields the following system of algebraic equations:

\[
0 = pA\left( v^2 - k^2 \right) B^2 (p - 1) - 2\left( a + bv^2 \right) pAB^4 \left\{ p^2 + (p - 2)^2 \right\} (p - 1) - cp^2 A^2 B^3 v(p - 1) + dp^2 A^2 B^4 v(p - 1)
\]

\[
-2p^2 A\left( v^2 - k^2 \right) B^2 + \left( a + bv^2 \right) pAB^4 \left\{ 4p^3 + (p - 1)^2 (p - 2) + (p + 1)^2 (p + 2) \right\} +
\]

\[
+ (c + d) p^2 A^2 B^4 v(3p - 1) = 0
\]

(13)

It is easy to see that equations (13) and (16) are an identity since \( p = 1 \). Substituting \( p = 1 \) in (14), (15) and (17) yields the following set of reduced equations:

\[
0 = p(p + 1)A\left( v^2 - k^2 \right) B^2 - 2\left( a + bv^2 \right) pAB^4 \left\{ p^2 + (p + 2)^2 \right\} (p + 1) - (c + d) p^2 A^2 B^4 v(3p + 1) = 0
\]

(14)

\[
0 = (a + bv^2) pAB^4 (p - 1)(p - 2)(p - 3) = 0
\]

(15)

\[
0 = \left( a + bv^2 \right) pAB^4 (p + 1)(p + 2)(p + 3) + (c + d) p^2 A^2 B^4 v(p + 1) = 0
\]

(16)

Now from (18)-(20), it is easy to observe that

\[
\frac{v^2 - k^2}{a + bv^2} = -4B^2 < 0
\]

(21)

This restriction between the parameters \( a, b, v \) and \( k \) must hold for the entire process to be true. Also, from (18) and (19), the velocity of the soliton is

\[
v = \frac{-(c + d) A + \sqrt{(c + d)^2 A^2 - 576abB^2}}{24bB}
\]

(22)

and the same expression of \( v \) is obtained by solving (20). Again, from (18) and (19),

\[
v = \sqrt{\frac{k^2 - 4abB^2}{1 + bB^2}}
\]

(23)
Now, equating the two values of the velocity of the soliton from (22) and (23) yields

\[
\left(1 + 4bAB^2\right)\left[(c + d)A - \sqrt{(c + d)^2 A^2 - 576abB^2}\right] = 576b^2 B^2 \left(k^2 - 4aAB^2\right)\]

which is the relation between the soliton free parameters \(A\) and \(B\). It is also easy to conclude from (22) and (24) that

\[
576abB^2 < (c + d)^2 A^2
\]

which is the restriction between the various parameters governing the BL equation.

Thus, finally, the topological 1-soliton solution, or the shock wave solution to the BL equation (1) is given by

\[
u(x, t) = A \tanh[B(x - vt)]
\]

where the relation between the free parameters \(A\) and \(B\) is given by (24), and the velocity of the soliton is given by (22) or (23). Also, the parameters as well as all coefficients of BL equation must obey the inequation given by (25) for the existence of solitons.

### 3. CONCLUSIONS

This paper obtains the mathematical structure of the topological 1-soliton solution of the BL equation. These solutions are going to be very helpful in carrying out further studies and analysis of the shock waves based on the BL model. For example, in (1+2) dimensions, where this BL equation is also commonly studied this solution can be mathematically structured to a generalized format. These solitons in (1+2) dimensions are also known as domain walls. The domain wall solutions of this equation in multi-dimensions will be obtained in future. Such results will be reported in a future publication.

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