SOLUTION OF THE RADIAL $N$-DIMENSIONAL SCHröDINGER EQUATION USING HOMOTOPY PERTURBATION METHOD

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Homotopy Perturbation Method (HPM) is applied to formulate analytic solution of the free particle radial dependent Schrödinger equation in $N$-dimensional space. This method is based on the construction of a homotopy with an embedding parameter $\delta \in [0, 1]$. The method shows its effectiveness, usefulness, and simplicity for obtaining approximate analytic solution. In addition, some interesting cases are analyzed and the effect of space dimension on the solution is pointed out.

Key words: Homotopy perturbation method, higher dimensions, approximation methods.

1. INTRODUCTION

The search for analytic exact or approximate solutions of different kinds of differential and integral equations has been of great importance over the years. Recently, a new analytic technique based on basic ideas of homotopy, called Homotopy Perturbation Method (HPM), was developed by He [1–4] and has been of great potential in solving different kinds of differential and integral equations in many areas of applied sciences and engineering. The method is a coupling between the traditional perturbation method and homotopy, which is a highly interesting and useful concept in topology, and deforms continuously to a simple problem which is easily solved. The HPM requires the construction of a homotopy with an embedding parameter $p \in [0, 1]$ which is considered as a small parameter whose range relates the initial solutions of a problem to its final solutions. If $p$ is 0, the problem reduces to a sufficiently simplified form, which normally admits a rather simple solution. As $p$ gradually increases to 1, the problem goes through a sequence of deformation. The final state of deformation is achieved when $p$ is 1 and the desired solution is then obtained. The method is considered as a summation of an infinite series which usually converges rapidly to the exact solution. The HPM

has been variously developed in different areas of applied sciences: It has been applied to different types of nonlinear problems [5-12], to integral equations [13-16], and to nonlinear differential equations with fractional time derivatives [17-21]. The HPM has also been used to find solutions to nonlinear Schrödinger equation [22-24] and to some kind of boundary value problems [25, 26]. Furthermore, the HPM was employed to find solutions to heat transfer problems and heat distribution for systems with variable thermal conductivity [27-30]. Over the last two decades, problems in the \( N \)-dimensional space are becoming increasingly important in different areas: For example, in quantum field theory [31], in quantum chemistry [32], in random walks [33], in Casimir effect [34], and in harmonic oscillators [35, 36]. Furthermore, the \( N \)-dimensional radial Schrödinger equation has been examined for different kind of potentials [37-39]. In the present paper, the HPM is proposed to provide approximate solutions for the free–particle radial part of Schrödinger equation in higher \( N \)-dimensional space. The outline of this paper is as follows: Section 2 gives details of He's homotopy perturbation method and shows how it can be applied to the \( N \)-dimensional Schrödinger equation. Section 3 presents details of application of HPM to our problem and the results obtained. Section 4 is devoted for conclusions.

2. HOMOTOPY PERTURBATION METHOD

To illustrate the basic ideas of the HPM, we consider the following nonlinear differential equation:

\[
A(u) - F(r) = 0, \quad r \in \Omega
\]  

with boundary conditions:

\[
B(u, \frac{du}{dn}) = 0, \quad r \in \Gamma
\]

Where \( A \) is a general differential operator, \( F(r) \) is a known analytic function, \( B \) is a boundary operator and \( \Gamma \) is the boundary of the domain \( \Omega \). The operator \( A \) can be divided into two parts: Linear operator \( L \) and nonlinear operator \( M \) so that Eq. (1) becomes

\[
L(u) + M(u) - F(r) = 0.
\]

Following He [1, 2], we construct a homotopy: \( V(r, p): \Omega \times [0, 1] \to R \) which satisfies

\[
H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - F(r)], \quad p \in [0, 1], \quad r \in \Omega
\]

or
Solution of the radial $N$-dimensional Schrödinger equation

$$H(v, p) = L(v) - L(u_0) + p[N(v) - F(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega$$  \hspace{1cm} (5)

Where $p \in [0, 1]$ is an embedding parameter, $u_0$ is an initial approximation of Eq. (1) which satisfies the boundary conditions. Eq.’s (4) and (5) give respectively:

$$H(v, 0) = L(v) - L(u_0) = 0$$  \hspace{1cm} (6)

$$H(v, 1) = A(v) - F(r) = 0$$  \hspace{1cm} (7)

The changing values of $p$ from zero to unity is just that of $V(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$, $A(v) - F(r)$ are called homotopic. Considering the embedding parameter $p$, as a small parameter, and applying the traditional perturbation technique, we can assume that the solution of Eq.(4) or (5) can be expanded as a power series in $p$; namely

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + ......$$  \hspace{1cm} (8)

Setting $p = 0$ yields the initial approximation $u_0(r)$, while $p = 1$ gives $u(r)$ as

$$u = \lim_{p \to 1} (u) = v_0 + v_1 + v_2 + v_3 + v_4 + ......$$  \hspace{1cm} (9)

3. FREE-PARTICLE $N$-DIMENSIONAL RADIAL SCHRÖDINGER EQUATION

The Laplacian operator in the $N$-dimensional spherical coordinates $(r, \theta_1, \theta_2, ..., \theta_{N-2}, \phi)$ has the form

$$\nabla^2 = r^{1-N} \frac{\partial}{\partial r} (r^{N-1} \frac{\partial}{\partial r}) + \frac{1}{r^2} \Lambda^2,$$  \hspace{1cm} (10)

where $\Lambda^2$ is a partial differential operator on the unit sphere $S^{N-1}$ given by [40]

$$\Lambda^2 = \sum_{k=1}^{N-2} \left( \prod_{j=1}^{k} \sin \theta_j \right)^{-2} \left( \sin \theta_k \right)^{k+1-N} \frac{\partial}{\partial \theta_k} \left( \sin \theta_k^{N-1} \frac{\partial}{\partial \theta_k} \right) + \left( \prod_{j=1}^{N-2} \sin \theta_j \right)^{-2} \frac{\partial}{\partial \phi}$$  \hspace{1cm} (11)

The separation of variables method separates the free-particle $N$-dimensional Schrödinger equation into two second-order differential equations [41]; namely

$$\Lambda^2 Y + \beta Y = 0$$  \hspace{1cm} (12)

$$\frac{d^2R}{dr^2} + \frac{(N-1)}{r} \frac{dR}{dr} + \left( k^2 - \frac{\beta}{r^2} \right) R = 0,$$  \hspace{1cm} (13)
where $k^2 = 2mE/h^2$ and $\beta$ is a separation constant whose values are given by [40]

$$\beta = \ell(\ell + N - 2).$$  \hfill (14)

The solutions to Eq.(12) are the hyper-spherical harmonics, $Y_{\ell}^{(m)}$, of degree $\ell$ on the unit sphere $S^{N-1}$. For each non-negative integer $\ell$, the number of hyper-spherical harmonics is given by [41]

$$n_{\ell} = \frac{(2\ell + N - 2)(\ell + N - 3)!}{\ell!(N - 2)!},$$  \hfill (15)

and are characterized by the integers $m_1$, $m_2$, $m_3$, ...., $m_{N-2}$ with the restrictions

$$\ell \geq m_1 \geq m_2 \geq m_3 \geq \ldots \geq |m_{N-2}| \geq 0.$$  \hfill (16)

The hyperspherical harmonics form an orthonormal set and thus they form a standard basis of the irreducible representations of the rotation group $SO(N)$ in the space of square integrable functions defined over the surface of the $N$-dimensional unit sphere with the invariant measure [40]

$$d\Omega = \prod_{j=1}^{N-2} \left(\sin \theta_j\right)^{N-1-j} d\theta_j d\phi.$$  \hfill (17)

In order to find the solution to the radial part given in Eq.(13), we consider the differential equation

$$\frac{d^2Y}{dx^2} + \left(\frac{1 - 2a}{x}\right) \frac{dY}{dx} + \left[\left(bcx^{-1}\right)^2 + \frac{(a^2 - p^2c^2)}{x^2}\right]Y = 0,$$  \hfill (18)

whose solution is given by [38]

$$Y(x) = x^a \left[ A J_{\rho} (bx^c) + B N_{\rho} (bx^c) \right],$$  \hfill (19)

where $J_{\rho}$ and $N_{\rho}$ are the ordinary Bessel and Neumann functions, respectively, and $a$, $b$, $c$ and $\rho$ are constants. Comparing Eq.(13) with Eq.(18) yields $b = k$, $c = 1$, $a = (2 - N)/2$, and $p = n + (N - 2)/2$. Therefore, with the use of Eq. (19), the solution to Eq. (13) can be immediately written down:

$$R(r) = r^{(2-N)/2} \left[ A J_{n+(N-2)/2} (kr) + B N_{n+(N-2)/2} (kr) \right],$$  \hfill (20)

where $A$ and $B$ are constants. The function $R(r)$ must be finite at $r = 0$ and thus the second term in Eq. (20) must be rejected due to the singular behavior of the Neumann function at the origin. Therefore, the solution in Eq. (20) reduces to
Solution of the radial $N$-dimensional Schrödinger equation

\[ R(r) = A r^{(2-N)/2} J_{+(N-2)/2}(kr), \]  
(21)

which can be written in terms of the spherical Bessel function as

\[ R(r) = A r^{(3-N)/2} j_{+(N-3)/2}(kr). \]  
(22)

The common representation of $j_m(kr)$ is given by

\[ j_m(kr) = \sum_{p=0}^{\infty} A_m \left( \frac{(-1)^p 2^n k^{2p} (m+p)!}{p! (2m+2p+1)!} r^{m+2p} \right), \]  
(23)

and thus Eq. (22) gives the series form of the radial solution $R(r)$, namely

\[ R(r) = \sum_{p=0}^{\infty} A_{N} \left( \frac{(-1)^p 2^{(N-3)/2} k^{2p} (\ell + p + (N-3)/2)!}{p! (2\ell + 2p + N - 2)!} r^{\ell+2p} \right). \]  
(24)

4. SOLUTION OF THE RADIAL PART OF THE FREE-PARTICLE'S SCHRODINGER EQUATION IN N DIMENSIONS USING HPM

In this section, the HPM is implemented, in an efficient way, to find approximate solutions for the radial part of Schrödinger equation in $N$-dimensional space subject to the condition that $R(r)$ is finite at $r=0$. Writing the Eq.(13) in the form

\[ r^{N-1} \frac{d^2 R}{dr^2} + (N-1)r^{N-2} \frac{dR}{dr} + \left[ k^2 r^{N-1} - \ell(\ell + N - 2)r^{N-3} \right] R = 0, \]  
(25)

and in view of Eq.(4) or (5), the homotopy for Eq.(25) can be constructed as

\[ H(R, p) = r^{N-1} \frac{d^2 R}{dr^2} + (N-1)r^{N-2} \frac{dR}{dr} + \left[ \frac{pk^2}{r^{1-N}} - \frac{\ell(\ell + N - 2)}{r^{3-N}} \right] R = 0, \]  
(26)

with $p \in [0, 1]$. The basic assumption of the HPM is that the solution $R(r)$ can be expressed as a power series in $p$, namely

\[ R(r) = \sum_{n=0} p^n R_n(r) = R_0(r) + pR_1(r) + p^2R_2(r) + p^3R_3(r) + .... \]  
(27)

Considering terms up to third power in the parameter $p$ and substituting Eq.(27) into Eq.(26) yields
\[ r^{N-1} \left[ R_0'' + pR_1'' + p^2R_2'' + p^3R_3'' \right] + \\
(N - 1)r^{N-2} \left[ R_0' + pR_1' + p^2R_2' + p^3R_3' \right] + \\
\left[ \frac{pk^2}{r_0^{N-1}} - \frac{\ell(\ell + N - 2)}{r_0^{N-3}} \right] [R_0 + pR_1 + p^2R_2 + p^3R_3] = 0. \]

(28)

Summing up the coefficients of equal power of \( p \) and setting each sum to zero, gives the following equations;

\[ p^0 : r^{N-1}R_0'' + (N - 1)r^{N-2}R_0' - \ell(\ell + N - 2)r^{N-3}R_0 = 0 \]

(29)

\[ p^1 : r^{N-1}R_1'' + (N - 1)r^{N-2}R_1' - \ell(\ell + N - 2)r^{N-3}R_1 = -k^2r^{N-1}R_0 \]

(30)

\[ p^2 : r^{N-1}R_2'' + (N - 1)r^{N-2}R_2' - \ell(\ell + N - 2)r^{N-3}R_2 = -k^2r^{N-1}R_1 \]

(31)

\[ p^3 : r^{N-1}R_3'' + (N - 1)r^{N-2}R_3' - \ell(\ell + N - 2)r^{N-3}R_3 = -k^2r^{N-1}R_2 \]

(32)

\[ p^n : r^{N-1}R_n'' + (N - 1)r^{N-2}R_n' - \ell(\ell + N - 2)r^{N-3}R_n = -k^2r^{N-1}R_{n-1} \]

(33)

where \( R_n(0) = R_n'(0) \) for \( n = 1, 2, 3, \ldots \).

In order to solve Equations (29-30), a series method can be used. This gives

\[ R_0(r) = C_0r^\ell + \frac{C_1}{r^{(\ell + N - 3)}} \], \hspace{1cm} (34)

Where \( C_0 \) and \( C_1 \) are constants. Since \( R(0) \) is finite, then the constant \( C_1 \) must be rejected and therefore Eq.(34) becomes

\[ R_0(r) = C_0r^\ell. \]

(35)

Substituting \( R_0(r) \) into Eq.(30) and again applying the series method, we get

\[ R_1(r) = \frac{-k^2C_0}{2(2\ell + N)}r^{(\ell + 2)}. \]

(36)

Similarly, the substitution of \( R_1(r) \) into Eq. (31) and solving the resulting equation gives

\[ R_2(r) = \frac{k^4C_0}{8(2\ell + N)(2\ell + N + 2)}r^{(\ell + 4)}. \]

(37)

A further substitution of \( R_2(r) \) into Eq.(32) and solving the resulting equation by series method yields
In general the $n^{th}$ term of the solution is given by

$$R_n(r) = C_0 \frac{(-1)^n k^{2n} (2 \ell + N - 2)! (\ell + n + (N - 3)/2)!}{(\ell + (N - 3)/2)! n! (2 \ell + N - 2 + 2n)!} r^{\ell + 2n}.$$  (39)

The above equation can be proved by mathematical induction as follows: For $n = 1$, Eq.(39) reduces to our result given in Eq.(36). Employing Eq.(33) for $n+1$ and using Eq.(39), we get

$$\frac{1}{p^{1-N}} R_{n+1}^{r_1} + \frac{N-1}{p^{2-N}} R_{n+1}^{r_2-N} = \frac{\ell}{p^{1-N}} R_{n+1}^{r_n} = -k^2 f(n) r^{N+2n+1},$$  (40)

where $f(n)$ is the coefficient of $R_n(r)$ given by Eq.(39). Assuming a series solution of the form

$$R_{n+1}(r) = \sum_j C_j r^j,$$  (41)

which upon its substitution into Eq.(40) gives

$$\sum_j j(j-1) + j(N-1) - \ell(\ell + N - 2) \left( r^{N+2n+1} = -k^2 f(n) r^{N+2n+1}. \right.$$  (42)

Equating powers of $r$ on both sides of Eq.(42) gives

$$j = \ell + 2n + 2,$$  (43)

and thus

$$-k^2 f(n) = C_j \left[ (\ell + 2n + 2)(\ell + 2n + 1) + (N-1)(\ell + 2n + 2) - \ell(\ell + N - 2) \right]
= C_j \left[ (\ell + 2n + 2)(\ell + 2n + N) + 2\ell - \ell(\ell + N) \right]
= C_j \left[ (\ell + 2n + 2)(\ell + N + 2n + 2) + 2\ell - \ell(\ell + N) \right]
= C_j \left[ (\ell + N)(2n + 2) + 2n(2n + 2) + 2\ell(n + 1) \right]
= 2 C_j \left[ (n+1)(2\ell + 2n + N) \right],$$  (44)

which gives

$$C_j = \frac{-k^2 f(n)}{2(n+1)(2\ell + 2n + N)}.$$  (45)

The substitution for $f(n)$ by the coefficient of $R_n(r)$ of Eq. (39) yields
Using the factorial property \((p+1)! = (p+1)p!\), we can write
\[
\frac{(\ell+n+(N-3)/2)!}{(\ell+n+(N-1)/2)!} = \frac{(\ell+n+(N-1)/2)!}{(\ell+n+(N-1)/2)!}
\]
\[
(\ell+2n+N-2)! (2\ell+2n+N) = \frac{(2\ell+2n+N)!}{(2\ell+2n+N-1)}
\]

The substitution of Eq. (47) into Eq. (46) and performing simple algebra, we get
\[
C_N = \frac{C_0 (-1)^{n+1} k^{2(n+1)} (2\ell + N - 2)! (\ell + n + (N-1)/2)!}{(\ell + (N-3)/2)! (n+1)! (2\ell + 2n + N)!}, \tag{48}
\]

and thus, with the help of Eq. (43), Eq. (41) immediately yields
\[
R_{n+1}(r) = \frac{C_0 (-1)^{n+1} k^{2(n+1)} (2\ell + N - 2)! (\ell + n + (N-1)/2)!}{(\ell + (N-3)/2)! (n+1)! (2\ell + 2n + N)!} r^{\ell+2n+2} \tag{49}
\]

It is clear to note that letting \(n \to n+1\) in Eq. (39) yields exactly the result in Eq. (49) and thus we proved our assertion in Eq. (39) by mathematical induction. In order to compare the solution predicted by the HPM, given by Eq. (39), with the exact solution given by Eq. (24), we let \(m = \frac{\ell + (N-3)/2}{2}\), so that
\[
\frac{(2\ell + N - 2)!}{(\ell + N-3/2)! (2\ell + 2n + N)!} = \frac{(2m+1)!}{m!} = 2^m (2m+1)!! \quad \text{and thus we have}
\]

The last step can be proved by mathematical induction as follows: It is trivial for \(m=1\), for \(m \to m+1\), we get
\[
\frac{(2m + 3)!}{(m+1)!} = \frac{(2m+3)(2m+2)(2m+1)!}{(m+1)m!}
\]

which, upon using Eq. (50), we get
\[
\frac{(2m + 3)!}{(m+1)!} = 2^{m+1} (2m+3)!! \quad \text{QED} \tag{51}
\]

Therefore, upon comparing Eq. (24) with Eq. (39), we get \(C_0 2^{(n(N-3)/2)} (2\ell + N - 2)!! = A_N 2^{(n(N-3)/2)}\), and thus we have
\[
A_N = C_0 (2\ell + N - 2)!! \tag{52}
\]
Therefore, the HPM solution yields the exact solution by choosing the arbitrary constant \( C_0 \) to satisfy Eq.(52).

It is worth to check the convergence of the solution just obtained by the HPM. One can check this by calculating the ratio \( \frac{R_{n+1}}{R_n} \), which can be computed from Eq.’s (39) and (49) with the result

\[
\frac{R_{n+1}}{R_n} = \frac{-k^2(\ell + n + 1 + (N - 3)/2)}{(n + 1)(2\ell + N + 2n)(2\ell + N + 2n - 1)} r^2.
\]

Using the identity \((p + 1)! = (p + 1)p!\), the above ratio becomes

\[
\frac{R_{n+1}}{R_n} = \frac{-k^2}{2(n + 1)(2\ell + N + 2n)} r^2.
\] (53)

It is noted that, for a given \( r, \ell, N \), this ratio decreases as \( 1/n^2 \).

5. SOME INTERESTING CASES

In this section, we analyze some interesting special cases that shed some light on the HPM solution.

5.1. THE CASE \( \ell = 0 \)

In this case, Eq.(39) yields

\[
R_n = \frac{C_0(-1)^n k^{2n} (N - 3)! (n + (N - 3)/2)!}{((N - 3)/2)! n!(2n + N - 2)!} r^{2n},
\] (54)

which immediately gives the first few terms, namely

\[
R_0 = C_0
\]

\[
R_1 = \frac{-C_0k^2}{2N} r^2
\]

\[
R_2 = \frac{C_0k^4}{2^22!N(N + 2)} r^4
\] (55)

\[
R_3 = \frac{-C_0k^6}{2^33!N(N + 2)(N + 4)} r^6
\]

It is interesting to write \( R \) up to the third term, namely
The extremes of $R$ can be found by setting $dR/dr = 0$ to get

$$r_{\text{min}} = \frac{1}{k} \sqrt{2(N+2)} ,$$  \hspace{1cm} (56)

and one can easily check that $d^2R/dr^2 > 0$ so that $R$ exhibits minimum at the above value of $r$. The value of this minimum is

$$R_{\text{min}} = \frac{C_0(N-2)}{2N}$$  \hspace{1cm} (57)

It is obvious that the minimum of $R$ increases with the space dimension $N$. It starts at $C_0/6$ in the three-dimensional space and $C_0/2$ in the infinite-dimensional space. It is tempting to consider the large $N$ limit where one may deduce from Eq. (55) that

$$R_n = \frac{C_0 (-1)^n k^{2n}}{2^n n! N^n} r^{2n} ,$$  \hspace{1cm} (58)

so that the solution could be written as

$$R = C_0 \sum \left( \frac{-k^2 r^2}{2N} \right)^n \frac{1}{n!} = C_0 \exp \left( \frac{-k^2 r^2}{2N} \right) ,$$  \hspace{1cm} (59)

which exhibits no extreme values.

5.2. THE CASE $\ell = 1$

Using Eq.(39), we get

$$R_0 = C_0 r$$

$$R_1 = -\frac{C_0 k^2}{2(N+2)} r^3$$  \hspace{1cm} (60)

$$R_2 = \frac{C_0 k^4}{2^2 2!(N+2)(N+4)} r^5$$

$$R_3 = -\frac{C_0 k^6}{2^3 3!(N+2)(N+4)(N+6)} r^7$$

For the large $N$ limit we have,
\[ R_n = C_0 \frac{(-1)^n k^{2n}}{2^n n! N^n} r^{2n+1}, \]

and thus the solution becomes

\[ R = C_0 r \sum_{n=0}^{\infty} \left( \frac{-k^2 r^2}{2N} \right)^n \frac{1}{n!} = C_0 r \exp\left(-k^2 r^2 / 2N\right) \] (61)

The extremes of the above solution can be found by setting \( dR/dr = 0 \) to get

\[ r_{\text{max}} = \frac{\sqrt{N}}{k} \] (62)

It is a simple matter to check that \( d^2 R / dr^2 \) is negative at \( r = r_{\text{max}} \) and thus \( R \) exhibits a maximum value, given by

\[ R_m = C_0 \frac{\sqrt{N}}{k} e^{-1/2} \] (63)

It is clear that as the space dimension increases, the position of the maximum and the value of the maximum shift towards higher values and in the infinite dimensional space \( (N \to \infty) \), both go to infinity.

6. CONCLUSION

In this work, the solution of the radial \( N \)-dimensional Schrödinger equation was obtained and analyzed using HPM. The obtained results show the effectiveness and usefulness of the homotopy perturbation method. It has been demonstrated that the solution obtained by HPM yields the exact solution by a proper choice of the arbitrary constant as seen in Eq.(52). For computational purposes, two special cases had been considered: For the case \( \ell = 0 \), we included the first three terms in the solution and it was found that the solution exhibits a minimum value of \( C_0 (N-2)/2N \) at \( r_{\text{max}} = \sqrt{2(N+2)} / k \). For the case \( \ell = 1 \), our results show that, for the large \( N \) limit, the series solution predicted by HPM has a closed form. This closed form has a maximum value of \( C_0 \sqrt{N} / ke \) at \( r_{\text{max}} = \sqrt{N} / k \) and both are increasing with the increase of the space dimension, \( N \). Furthermore, in the infinite-dimensional space, for this second case, the maximum value becomes infinite at infinity. Therefore, this work demonstrates the powerful of the HPM and illustrates the effect of the space dimension on the solution.
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