In this work, we focus on the dynamics of Dirac particles in the presence of a constant electric field in a cosmological anisotropic universe. Instead of Einstein’s theory of general relativity, we perform the calculations using the teleparallel theory of gravity which is also called as the torsion gravity. First, we found the exact solution of the teleparallel Dirac equation in an anisotropic Bianchi-I universe. Second, the harmonic oscillator behaviour of the solution and then the quantization of oscillation frequency had been discussed. Third, we investigated the spin precession of Dirac particles and dispute the axial-vector spin coupling term.

Key words: Dirac; spin-1/2 particle; torsion; teleparallel gravity.

PACS: 04.20.-q; 04.20.Gz.

1. INTRODUCTION

The general three parameter theory (namely the teleparallel gravity) introduced first in 1979 by Hayashi and Shirafuji [1]. The teleparallel equivalent of general relativity can be handled as a gauge theory for the translation group based on Weitzenböck geometry [2, 3].

The gravitational interaction, in the teleparallel theory of gravity, is described by a force similar to the Lorentz force of electrodynamics with torsion playing the role of force and with identically vanishing curvature tensor [4]. Instead of the metric tensor, the basic entity of teleparallel gravity is the non-trivial tetrad field (in general relativity the metric tensor plays the role of the basic entity). These two theories provide equivalent descriptions of the gravitational interaction and shows that torsion and curvature might be simply two alternative ways of describing the gravitational field (the source of curvature in general relativity and the source of torsion in teleparallel gravity) [5, 6].

The tetrad formalism has some advantages which come mainly from its independence from the equivalence principle, and consequently it is suitable for the discussion of quantum issues [6].
In the presence of electromagnetic fields the Teleparallel Dirac Equation (TDE) is written as:

\[
\{ \gamma^\mu (\partial_\mu - \Gamma^\mu - ieA_\mu) + m \} \Psi = 0, \tag{1}
\]

where \( \gamma^\mu \) are the curved Dirac matrices and their connection with \( \tilde{\gamma}^\mu \) (Minkowski-Dirac matrices) are established by tetrad fields \( h^\mu_i \) [7]. The dynamical effects of spacetime on the spin is brought into the Dirac equation through the spin connection coming into sight in the Dirac equation including gravitation [7,8]. In the teleparallel theory of gravity, the spinor connections are defined as:

\[
\Gamma_\mu = \frac{1}{2} V_\mu - \frac{3i}{4} B_\mu \tilde{\gamma}_5. \tag{2}
\]

Here, \( V_\mu \) and \( B_\mu \) are the vector part and the axial-vector part of the torsion tensor, respectively. The torsion tensor is divided into three irreducible parts under the global Lorentz transformation group [8]. These are the tensor, the vector and the axial-vector parts. Hence, the tensor part is given by the following relation:

\[
t_{\alpha\mu\nu} = \frac{1}{2}(T_{\alpha\mu\nu} + T_{\alpha\nu\mu}) + \frac{1}{6}(g_{\nu\alpha} T_{\delta\mu}^\delta + g_{\nu\mu} T_{\omega\alpha}^\omega) - \frac{1}{3} g_{\alpha\mu} T^\rho_{\rho\nu}, \tag{3}
\]

and the vector and the axial-vector parts are defined by

\[
V_\mu = T^\alpha_{\alpha\mu}, \tag{4}
\]
\[
B_\mu = \frac{1}{6} \varepsilon^{\mu\nu\alpha\beta} T_{\nu\alpha\beta}. \tag{5}
\]

Now, the torsion tensor is described by using these three components as follows:

\[
T_{\alpha\mu\nu} = \frac{1}{2}(t_{\alpha\mu\nu} - t_{\alpha\nu\mu}) + \frac{1}{3}(g_{\alpha\mu} V_\nu - g_{\alpha\nu} V_\mu) + \delta_{\alpha\mu\sigma} B^\sigma, \tag{6}
\]

where

\[
\delta^{\alpha\mu\sigma} = \frac{1}{\sqrt{-g}} \varepsilon^{\alpha\mu\sigma}. \tag{7}
\]

It is important to mention here that the deviation is described by the axial-vector torsion. Next, the definition of the Weitzenböck connection [2] is

\[
\Gamma^\lambda_{\alpha\beta} = \tilde{\Gamma}^\lambda_{\alpha\beta} - \Upsilon^\lambda_{\alpha\beta}, \tag{8}
\]

where \( \tilde{\Gamma}^\lambda_{\alpha\beta} \) shows the Levi-Civita connection of the metric \( g_{\alpha\beta} = \eta_{ij} h^i_{\alpha} h^j_{\beta} \), and is defined by

\[
\tilde{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}), \tag{9}
\]

and

\[
\Upsilon^\lambda_{\alpha\beta} = \frac{1}{2} \left( T^\lambda_{\alpha\beta} + T^\lambda_{\beta\alpha} - T^\lambda_{\alpha\beta} \right) \tag{10}
\]

gives the contortion tensor. Here,

\[
T^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha} - \Gamma^\lambda_{\alpha\beta} \tag{11}
\]
is the torsion of the Weitzenböck connection. A non-trivial field can be considered to represent the linear Weitzenböck connection [9]

\[ \Gamma^\lambda_{\alpha\beta} = h^i_\lambda \partial_\beta h^i_\alpha. \] (12)

By using a tetrad field satisfying

\[ h^i_\alpha h^i_\beta = \delta^\beta_\alpha, \quad h^i_\alpha h^\alpha_j = \delta^i_j. \] (13)

the Tensor and Lorentz indices can be interchanged.

In order to denote the tensor indices in relation with spacetime the Greek alphabet will be used and to denote the local Lorentz indices the Latin alphabet will be used.

2. THE EXACT SOLUTION OF TDE

In the teleparallel theory of gravity, we focus on an anisotropic universe which is defined by the following line-element,

\[ g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + t^2(dx^2 + dy^2) + dz^2. \] (14)

At \( t = 0 \), the metric given above has a space-like singularity. We write TDE for this space-time by taking into account the following vector potential [10],

\[ A_\mu = (0, 0, 0, -Et). \] (15)

Note that \( A_\mu = (A_0, \vec{A}) \). One can easily find out that

\[ F^{\mu\nu} F_{\mu\nu} = -2E^2, \quad \hat{F}^{\mu\nu} F_{\mu\nu} = 0 \] (16)

denote the absence of magnetic field [10]. Using the line-element given above, we obtain the following results for the metric tensor and its inverse:

\[ g^{\mu
u} = -\xi^{00} + \xi^{11} t^2 + \xi^{22} t^2 + \xi^{33}, \] (17)

\[ g^{\mu\nu} = -\xi_{(0)}^{\mu\nu} + \frac{1}{t^2} \xi_{11}^{\mu\nu} + \frac{1}{t^2} \xi_{22}^{\mu\nu} + \xi_{33}^{\mu\nu}, \] (18)

where \( \xi^{\alpha\beta}_\mu = \delta^\alpha_\mu \delta^\beta_\nu \). Next, we choose to work in the following tetrad fields:

\[ h^i_\mu = \delta^{(0)}_i \delta^0_\mu + t \delta^{(1)}_i \delta^1_\mu + t^2 \delta^{(2)}_i \delta^2_\mu + t^3 \delta^{(3)}_i \delta^3_\mu, \] (19)

\[ h^i_\mu = \delta^{(0)}_i \delta^0_\mu + \frac{1}{t} \delta^{(1)}_i \delta^1_\mu + \frac{1}{t^2} \delta^{(2)}_i \delta^2_\mu + \delta^{(3)}_i \delta^3_\mu. \] (20)

For the Weitzenböck connections, the non-vanishing components in this space-time are obtained as follows:

\[ \Gamma^1_{10} = \Gamma^2_{20} = \frac{1}{t}. \] (21)
Hence, the surviving components of the torsion tensor are
\[ T_{101}^1 = T_{02}^2 = \frac{1}{t}, \]
(22)
\[ T_{110}^1 = T_{220}^2 = -\frac{1}{t}. \]
(23)
If one use the results given above in the eqn. (4), the following components of the vector torsion are found:
\[ V_0 = -\frac{2}{t}, \quad V_1 = V_2 = V_3 = 0. \]
(24)
Therefore, we find that there is no any non-zero components of the axial-vector torsion by using eqn. (5):
\[ B_0 = \vec{B} = 0. \]
(25)
By taking into account the results given in eqns. (4) and (5), one can write the spin connections which help one to define TDE. The corresponding components are:
\[ \Gamma_0 = -\frac{1}{t}, \quad \Gamma_1 = \Gamma_2 = \Gamma_3 = 0. \]
(26)
Nevertheless, the TDE takes the following form:
\[ \left[ \tilde{\gamma}_0 \left( \frac{\partial}{\partial t} + \frac{1}{t} \tilde{\gamma}_1 \frac{\partial}{\partial x} + \tilde{\gamma}_2 \frac{\partial}{\partial y} + \tilde{\gamma}_3 \frac{\partial}{\partial z} + i e E t \right) + m \tilde{\gamma}_3 \tilde{\gamma}_0 \right] \tilde{\gamma}_3 \tilde{\gamma}_0 \Psi(t, x, y, z) = 0 \]
(27)
Now, let us introduce \( \Psi = t^{-1} \Xi \) in order to cancel the contribution due to the zeroth-component of spinor connections. Then, we get
\[ \left\{ \tilde{\gamma}_0 \frac{\partial}{\partial t} + \frac{1}{t} \left( \tilde{\gamma}_1 \frac{\partial}{\partial x} + \tilde{\gamma}_2 \frac{\partial}{\partial y} + \tilde{\gamma}_3 \frac{\partial}{\partial z} + i e E t \right) + m \tilde{\gamma}_3 \tilde{\gamma}_0 \right\} \tilde{\gamma}_3 \tilde{\gamma}_0 \Xi = 0. \]
(28)
We know that \( (\tilde{\gamma}_0)^2 (\tilde{\gamma}_3)^2 = -1 \). After some mathematical steps, we find \( \tilde{\gamma}_3 \tilde{\gamma}_0 \tilde{\gamma}_3 \tilde{\gamma}_0 = 1 \). Hence, we can transform eqn. (28) into the following form:
\[ \left\{ \tilde{\gamma}_3 \frac{\partial}{\partial t} + \frac{1}{t} \left( \tilde{\gamma}_1 \frac{\partial}{\partial x} + \tilde{\gamma}_2 \frac{\partial}{\partial y} + \tilde{\gamma}_3 \frac{\partial}{\partial z} + i e E t \right) + m \tilde{\gamma}_3 \tilde{\gamma}_0 \right\} \tilde{\gamma}_3 \tilde{\gamma}_0 \Xi = 0. \]
(29)
One can now write equation (29) as a sum of two first-order commuting differential operators \[11]:
\[ (\hat{L}_1 + \hat{L}_2) \Phi(t, x, y, z) = 0 \]
(30)
where,
\[ \hat{L}_1 = t \left( \tilde{\gamma}_3 \frac{\partial}{\partial t} + \tilde{\gamma}_0 \left[ \frac{\partial}{\partial z} + i e E t \right] + m \tilde{\gamma}_3 \tilde{\gamma}_0 \right), \]
(31)
\[ \hat{L}_2 = \left[ \tilde{\gamma}_1 \frac{\partial}{\partial x} + \tilde{\gamma}_2 \frac{\partial}{\partial y} \right] \tilde{\gamma}_3 \tilde{\gamma}_0, \]
(32)
\[ \Phi = \tilde{\gamma}_3 \tilde{\gamma}_0 \Xi. \]
(33)
By defining a separation constant $\lambda$, eqn. (30) can be rewritten as follows:

$$\hat{L}_1 \Phi = -\lambda \Phi,$$  \hspace{5pt} (34)

$$\hat{L}_2 \Phi = \lambda \Phi.$$ \hspace{5pt} (35)

We see that eqn. (30) commutes with $\vec{\nabla} = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}$, nonetheless, the spinor $\Phi$ can be written as

$$\Phi = e^{i(k_1 x + k_2 y + k_3 z)} \chi(t).$$ \hspace{5pt} (36)

We prefer to do required calculations by using the following representation of the Dirac matrices:

$$\tilde{\gamma}^0 = \begin{pmatrix} -i \sigma_1 & 0 \\ 0 & i \sigma_1 \end{pmatrix}, \quad \tilde{\gamma}^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \tilde{\gamma}^2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad \tilde{\gamma}^3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}.$$ \hspace{5pt} (37)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ \hspace{5pt} (38)

Using these representations and the definition

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$ \hspace{5pt} (40)

in eqn. (35), one can find an algebraic result which gives a chance to determine the relation between to components of the bispinor $\chi$:

$$\chi_2 = \frac{k_1 \sigma_2}{ik_2 + \lambda \chi_1}.$$ \hspace{5pt} (41)

Here, the eigenvalue is $\lambda = i \sqrt{k_1^2 + k_2^2}$.

Next, we will focus on eqn. (34) while considering the representations introduced above. Using the result given in eqn. (41), we find that, for $k_3 = 0$, eqn. (34) is reduced to the following system of equations:

$$\frac{d\varphi_1}{dt} + \frac{\lambda}{t} \varphi_1 + (im + eEt)\varphi_2 = 0,$$ \hspace{5pt} (42)

$$\frac{d\varphi_2}{dt} + \frac{\lambda}{t} \varphi_2 - (im - eEt)\varphi_1 = 0.$$ \hspace{5pt} (43)

Here, we defined $\chi_1 = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$. From eqns. (42) and (43), we find the following second-order equation:

$$\frac{1}{t} \frac{d\varphi_2}{dt} + \frac{\lambda(\lambda - 2)}{t^2} \varphi_2 - \frac{d^2\varphi_2}{dt^2} - (m^2 + e^2E^2t^2)\varphi_2 = 0$$ \hspace{5pt} (44)
While obtaining this equation, we have neglected the mass in the first-order variation term of $\varphi_2$. Now, let us introduce $\varphi_2 = \frac{t^\frac{3}{2}}{}$ in eqn. (44), then we have

$$\frac{d^2 \Im}{dt^2} + \left( -\frac{\lambda^2 - 2\lambda + \frac{3}{4} t^2}{t^2} + m^2 + e^2 E^2 t^2 \right) \Im = 0. \quad (45)$$

One can transform this equation into the well-known Whittaker Equation by defining $u = i e E t^2$ and $\Im = u^{-\frac{1}{4}}\Theta$. From this point of view, the solution of eqn. (44) can be expressed in terms of Whittaker functions as

$$\varphi_2 = (i e E t)^{-\frac{1}{4}} \left[ \alpha_1 M_{\zeta,\rho}(i e E t^2) + \alpha_2 W_{\zeta,\rho}(i e E t^2) \right], \quad (46)$$

where, $\alpha_1$ and $\alpha_2$ are arbitrary constants, $\zeta = -\frac{im^2}{4e}$ and $\rho = \frac{\lambda - 1}{2}$. On the other hand, one can obtain the following result of $\varphi_1$ in a similar way:

$$\varphi_1 = (i e E t)^{-\frac{1}{4}} \left[ \alpha_1 M_{\zeta,\rho^+}(i e E t^2) + \alpha_2 W_{\zeta,\rho^+}(i e E t^2) \right], \quad (47)$$

where $\rho^+ = \frac{\lambda + 1}{2}$. In the general case, the exact solution of TDE can be written in the form:

$$\Psi(t, x, y, z) = -\frac{e^{i(k_1 x + k_2 y + k_3 z)}}{t^\frac{1}{4} i e E} \left( \begin{array}{c} -i \varphi_2 \\ i \varphi_1 \\ \frac{k_1}{ik_2 + \lambda} \varphi_1 \\ \frac{k_3}{ik_2 + \lambda} \varphi_2 \end{array} \right). \quad (48)$$

3. QUANTIZATION OF THE OSCILLATION FREQUENCY

One of the ways to obtain the frequency spectrum is considering the circumstance on solutions of the differential equation. The solutions found must be bounded for all the values as done in quantum mechanics. This method gives the quantization of frequency. The line-element we considered describes an expanding model, nevertheless, we might expect to obtain gravitational red-shift in the frequency.

We can write eqn. (45) as follows:

$$\Im''(t) + \omega^2 \Im(t) = 0. \quad (49)$$

Then, the frequency $\omega$ is defined by

$$\omega^2 = -\frac{\lambda^2 - 2\lambda + \frac{3}{4} t^2}{t^2} + m^2 + e^2 E^2 t^2. \quad (50)$$

Nevertheless, the oscillation region [12] is given as below

$$\omega^2 - m^2 - \beta < 2 e^2 E^2 t^2 < \omega^2 - m^2 + \beta, \quad (51)$$

here we introduced

$$\beta^2 = (\omega^2 - m^2)^2 + \frac{m^4}{4} + 2ieEm^2 + 3e^2 E^2. \quad (52)$$
From the condition on Whittaker functions that should be bounded for all values of the variable,
\[ \zeta - \rho^2 + \frac{1}{2} = -n, \]  
we obtain the quantization for the spacetime given in eqn. (14) as follows:
\[ k_\perp = i \left[ 2n - 1 - \frac{im^2}{2eE} \right], \]  
here, \( n \) is a positive integer or zero and we define \( \lambda = ik_\perp \).

4. THE SPIN PRECESSION

Many researchers [3, 13–18] showed that the spin precession of a Dirac particle is connected with the axial-vector torsion, and it will be engrossing to perceive that the axial-vector torsion designate the deviation of the axial symmetry from the spherical symmetry [14].

\[ \frac{d\vec{S}}{dt} = -\frac{3}{2} \vec{B} \times \vec{S} \]  
where \( \vec{S} \) is the semi-classical spin vector of a Dirac particle and \( \vec{B} \) is the space-like part of the axial-vector torsion. Hence, the coincident additional Hamiltonian term is
\[ \delta H = -\frac{3}{4} \vec{B} \cdot \vec{\sigma} \]  
where \( \vec{\sigma} \) represents the spin of the particle and is defined by \( \vec{S} = \frac{\vec{\sigma}}{2} \) [19]. In section 2, we calculate that the axial-vector part of torsion vanishes,
\[ B^\mu(t, x, y, z) = 0. \]  
However, in space-like vector form, the axial-vector becomes
\[ \vec{B}(t, x, y, z) = 0. \]  
Then, we arrive at the spin vector of the Dirac particle is constant, and the corresponding Hamiltonian induced by the axial-vector spin coupling vanishes. Since the torsion takes part as the gravitational force in teleparallel gravity, a spinless particle will obey the force equation [9, 18] in the gravitational field
\[ \frac{du_\lambda}{ds} - \Gamma_{\mu\lambda\nu} u^\mu u^\nu = T_{\mu\lambda\nu} u^\mu u^\nu. \]  
The left side defines the Weitzenböck covariant derivative of \( u_\lambda \) along the world-line of the particle. The presence of the torsion tensor on its right hand side means torsion gives the role of an external force in teleparallel theory of gravity.
5. FINAL REMARKS

We have analysed the Dirac equation in the background of spacetime based on an anisotropic Bianchi-I type metric. We found the exact solutions of the Dirac equation in the teleparallel gravity, obtained the oscillating region for the Dirac particles, quantized the oscillation frequency, and disputed the spin precession. Furthermore, the result emphasizes the importance of the teleparallel gravity. Hence, it motivates us to use the teleparallel theory to investigate other gravitational problems. The other motivation is that these results can be used to discuss quantum field theory in curved expanding spacetimes.

REFERENCES

2. R. Weitzenbock, Invarianten Theorie (Gronningen, Noordhoff, 1923).