STUDY OF THE VLIJENGHART’S STABILITY CONDITION OF THE FD
SCHEMES SIMULATING THE KdV SOLITONS PROPAGATION

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As it is known, one of the most important present research topics is that of the gradual
structure degradation (de-structuration) processes of some organised pulses (as those
simulating the technical solitons, ageing people, financial banks, etc). Given being
these processes are due to the distortions accumulation through discrete non-linear
interactions, one of the most suitable tools to study them is the FD simulation of the
KdV solitons propagation. Our studies [13] pointed out that frequently the FD time
steps $\Delta t$ can be somewhat larger than the threshold indicated by the classical
Vliegenthart’s stability condition [10]: $\Delta t \leq A_s \frac{4\mu}{(A_s)^2} \cdot \epsilon^{-1}$. This work
thoroughly studies the possible approximations of the more general von Neumann’s
type specific condition corresponding to the stability of FD schemes of the KdV
solitons propagation simulation:

$$
\sin\xi \frac{\Delta \xi}{\xi} \left[ e \cdot u - \frac{\mu}{(\Delta s)^2} (1 - \cos\xi) \right] \leq 1,
$$

where $\xi = k \cdot \Delta s$ ($k \in N$).

Key words: Computer simulations, FD schemes, KdV solitons, Stability, Vliegenthart’s
condition.

1. INTRODUCTION

As it is well known, the evolving high complexity systems (e.g. the physical
solitons, the living beings, swarms, financial markets, etc) have a specific average
lifetime. Even from the discovery [1] of the simplest such systems – the solitons, it
was found the gradual degradation of their structure (confirmed also for the optical
solitons [2]), due to the interactions with the non-homogeneous propagation medium.

That is why the derivation of the solitons structure in ideal media, by means of the classical methods of Ablowitz [3], Hirota [4], Mihalache [5], etc, represent only the first step in the numerical evaluation of the average lifetime of certain solitons. We will mention also that recently was found that the study of non-classical symmetries of the nonlinear evolution equations generates also [6] some solutions of certain rather intricate such equations, as the one-dimensional Robinson-Trautman one:

$$u_t = -\frac{u_{xxx}}{u^2} + \frac{6u_{xxx}u_x}{u^3} + \frac{4u_{xx}^2}{u^3} - 21\frac{u_{xx}u_x^2}{u^4} + 12\frac{u_x^4}{u^5}. \quad (1)$$

Taking into account the usual discrete (local) character of the interactions with the surroundings, the most suitable computation method is that of the Local Interaction Simulation Approach (LISA) [7], which uses the Finite-Differences (FD) descriptions and simulations.

After the evaluation of the stability radius (the number $N_{\text{stab}}$ of FD steps up to the instability installation) [8], the specific average life-time can be evaluated by means of the elementary relation: $t_{\text{life}} = N_{\text{stab}} \cdot \Delta t_{\text{max}}$, where $\Delta t_{\text{max}}$ is the maximum value of the time step corresponding to a given space step $\Delta s$.

The derivation of the expression of $\Delta t_{\text{max}}$ can be achieved by means of the classical von Neumann stability criterion [9]. For the classical Korteweg-de Vries (KdV) solitons [10] a preliminary expression of the stability condition (and of the $\Delta t_{\text{max}}$) was derived by Vliegenthart [11], his expression being still used in different scientific works and monographs [12]. We have to underline also the possibilities of tsunami waves modeling starting from the forced KdV equation, subject to a similar Vliegenthart’s stability condition [11c].

Given being: a) the rather high interest of the evaluations of the physical solitons average life-time, b) some of our previously accomplished calculations [13], which have indicated some significant deviations from the Vliegenthart’s expression, the aim of this study is to examine rigorously the FD stability criterion for the FD simulations of the KdV solitons propagation.

2. BASIC NOTIONS

One of the most unpleasant numerical phenomena met in the frame of the Finite Differences (FD) simulations of pulses propagation is that of instability [8, 14, 16, 17]. As it is known, in certain conditions these FD schemes enter a regime
of continuous amplification of some local random errors, leading to completely erroneous unlimited values of the pulse components (the instability). To avoid the instability numerical phenomena, the parameters of the FD simulations (the FD time $\Delta t$ and space $\Delta s$ steps) have to be chosen to fulfill the stability conditions, specific to each propagation equation.

In the frame of his work [11], Vliegenthart examined the stability conditions of the FD schemes of several equations, of the Korteweg-de Vries one, particularly. Both the Vliegenthart’s KdV stability condition, that corresponding to the Boussinesq equation [9b], etc are based on the von Neumann analysis [9a] of the FD schemes, using the Fourier expansion:

$$u_j^n = \sum_{k=-\infty}^{\infty} C_k \cdot g(k)^n \cdot \exp(ik \cdot j \Delta s), \quad (2)$$

of the exact solution $u_j^n = u(n \cdot \Delta t, j \cdot \Delta s)$ of the pulse propagation equation.

In this manner, Vliegenthart obtained [11] the FD schemes stability conditions for KdV solitons propagation simulations as:

$$\frac{\Delta t}{\Delta s} \left[ \varepsilon |u| + 4 \frac{\mu}{(\Delta s)^2} \right] \leq 1, \quad (3)$$

where $\varepsilon$ and $\mu$ are the non-linearity and the dispersion coefficients, respectively, from the expression:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \varepsilon \cdot u \frac{\partial u}{\partial x} + \mu \frac{\partial^3 u}{\partial x^3} = u_t + u_x + \varepsilon \cdot u \cdot u_x + \mu \cdot u_{xxx} = 0 \quad (4)$$

of the KdV equation. The expression of the KdV equation can be simplified as:

$$u_t + \varepsilon \cdot u \cdot u_x + \mu \cdot u_{xxx} = 0, \quad (5)$$

by means of the substitution: $s = x - t$.

Given being: a) despite the frequent use of Vliegenthart’s stability condition in many (even recent) numerical simulations of the KdV solitons propagation (e.g. [12], particularly), its derivation was not too rigorous, b) our studies [13] pointed out that this condition is a sufficient one, but not also a strictly necessary condition, the main goals of this work are to: (i) find why the Vliegenthart’s condition does not is also a strictly necessary stability condition, (ii) identify some more accurate approximations of the stability condition of the FD schemes intended to the KdV solitons propagation simulation.
3. RIGOROUS DERIVATION OF THE STABILITY CONDITION OF THE 2 STEPS DISCRETIZED FD SCHEMES OF THE PROPAGATION EQUATION (5) OF KdV SOLITONS

Substituting the Fourier expansion (2) of the exact solution $u_j^n = u(n \cdot \Delta t, j \cdot \Delta s)$ in the classical 2 steps FD symmetric discretization of the propagation equation (5) of KdV solitons:

$$\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta t} + \varepsilon \cdot u \cdot \frac{u_{j+1}^n - u_j^n}{\Delta s} + \mu \cdot \frac{u_{j+2}^n - 2u_{j+1}^n + 2u_j^n - u_{j-2}^n}{(\Delta s)^3} = 0.$$ (6)

one obtains the characteristic equation of the amplification factors $g_k(k \cdot \Delta s)$ corresponding to the studied FD scheme:

$$g_+^2 + 2i \cdot b \cdot g_+ - 1 = 0,$$ (7)

where:

$$b = \sin \xi \cdot \frac{\Delta t}{\Delta s} \left[ \varepsilon \cdot u - 2 \frac{\mu}{(\Delta s)^2} (1 - \cos \xi) \right], \text{ with } \xi = k \cdot \Delta s.$$ (8)

The solutions of the characteristic equation (7) are: $g_+ = -i \cdot b \pm \sqrt{1 - b^2}$. Given being: $g_+ \cdot g_- = -1$, the product $|g_+| \cdot |g_-| = 1$, hence if $|b| > 1$ it results that $|g_-| > 1$ and the studied FD scheme is unstable. To have both $|g_+|, |g_-| \leq 1$, it is necessary that $b = \sin \theta$, hence: $g_- = -e^{i\theta}$ and $g_+ = e^{-i\theta}$. In this manner, both the stability conditions: $|g_\pm| \leq 1$ and the relation: $g_+ \cdot g_- = -1$ are fulfilled if:

$$|b| \leq \left| \sin \xi \cdot \frac{\Delta t}{\Delta s} \left[ \varepsilon \cdot u - 2 \frac{\mu}{(\Delta s)^2} (1 - \cos \xi) \right] \right| \leq 1.$$ (9)

Using the symbol: $x = \sin \xi$, and observing that the minimum value of the threshold $\Delta t_{\text{max}}(x, \Delta s; \varepsilon, \mu, u_{\text{max}})$ of the time step $\Delta t$ is reached for negative values $\cos \xi = -\sqrt{1 - x^2}$, one finds that the stability condition (9) can be written in the equivalent form:

$$\Delta t \leq \min \left[ \frac{\Delta s}{x \left[ \varepsilon \cdot u - 2\mu \left( 1 + \sqrt{1 - x^2} \right) / (\Delta s)^2 \right]} \right] = \Delta t_{\text{max}},$$ (10)
As it is well-known, the exact solution of the propagation equation (5) of KdV solitons leads only to positive values of all displacements $u$. If the considered FD schemes are not too accurate (e.g., if they don’t consider the effects of the boundary conditions, etc), they produce after several time steps some negative local values: $u^n_j = u(n \cdot \Delta t, j \cdot \Delta x) < 0$ of the displacement. To ensure the stability of the considered FD scheme in such cases, the condition (10) has to be written as:

$$\Delta t \leq \min \left\{ \frac{\Delta x}{\sin \xi \cdot \left[ \varepsilon \cdot |u| + 2\mu (1 - \cos \xi) / (\Delta x)^2 \right]} \right\}. \quad (10')$$

Because $\sin^2 \xi + \cos^2 \xi = 1$, it is not possible to have concomitantly $\cos \xi = -1$ and $\sin \xi = \pm 1$. To ensure the stability of the FD schemes intended to the description of KdV solitons propagation, Vliegenthart’s renounced to the search of the exact value of $\xi$ which leads to the minimum of the ratio from the right part of expression (10’) and has written the stability condition in the form:

$$\Delta t \leq \frac{\Delta x}{\varepsilon \cdot |u| + 4\mu / (\Delta x)^2} = \Delta t_{\text{Vli}}, \quad (11)$$

which is obviously a sufficient condition, but not the exact necessary stability condition.

4. SEARCH OF A MORE ACCURATE APPROXIMATION OF THE STABILITY CONDITION OF THE 2 STEPS DISCRETIZED FD SCHEMES OF KdV SOLITONS MOTION

For sufficiently accurate FD schemes, the displacements $u$ keep their initial positive values and their maximum value will correspond to the soliton amplitude $A$. For this reason, we will study numerically in following the function:

$$f(x, \Delta x) = \frac{\Delta x}{x \left[ \varepsilon \cdot A - 2\mu \left(1 + \sqrt{1 - x^2} \right) / (\Delta x)^2 \right]}, \quad (12)$$

involved by the stability condition (10).

Given being this function depends additionally on the numerical values of the parameters $\varepsilon$ (non-linearity), $\mu$ (dispersion coefficient) and of the amplitude $A$, we
studied previously the values of these parameters used in the frame of some already accomplished FD numerical simulations. The resulted findings are synthesized by Table 1.

Table 1

<table>
<thead>
<tr>
<th>Work</th>
<th>$\varepsilon$</th>
<th>$\mu$</th>
<th>$A$</th>
<th>$\Delta s$</th>
<th>$\Delta t$</th>
<th>Mentions</th>
</tr>
</thead>
<tbody>
<tr>
<td>[8]</td>
<td>0.2</td>
<td>0.1</td>
<td>1</td>
<td>0.4</td>
<td>0.1</td>
<td>$A =</td>
</tr>
<tr>
<td>[9], [10]</td>
<td>0.04558</td>
<td>0.000121</td>
<td>1</td>
<td>0.25</td>
<td>0.2194</td>
<td>Chosen small values of $\varepsilon$ and $\mu$ to study many iterations</td>
</tr>
<tr>
<td>[1]</td>
<td>0.1</td>
<td>0.2</td>
<td>2</td>
<td>0.05 … 0.5</td>
<td>Various</td>
<td>Stable runs for $\Delta t$ up to 50% larger than $\Delta t_{Vli}$ [Vliegenthart’s threshold (11)]</td>
</tr>
</tbody>
</table>

Taking into account that the used numerical values of the main parameters of the KdV solitons simulations are generally of the same magnitude order, we used in following the values of works [13].

Because the $\sqrt{1 - x^2} = f(x)$ dependence is rather weak for $x < 0.5$, one finds from relation (12) that for such values $f(x, \Delta s) = \frac{\Delta s}{x \left[ \varepsilon \cdot A - 4\mu/(\Delta s)^2 \right]}$, hence the corresponding double-logarithmic plots have to be (approximately) linear. The analysis of relation (10), points out the function $f(x, \Delta s)$ begins to increase only in the proximity of 1, when $x \approx$ constant, and $\sqrt{1 - x^2}$ is quickly decreasing.

For these reasons, the Figures 1 and 2 present both the usual “linear” and the double-logarithmic plots of the $f(x)$ dependencies, respectively for the chosen values $x = 0.05; 0.1; 0.2$ and $0.4.$
The obtained numerical results referring to the thresholds $\Delta t_{\text{max}}$ and $\Delta t_{\text{Vli}}$ of the FD time step corresponding to the rigorous stability condition (10) and to the Vliegenthart’s approximation (10) for different values of the space step $\Delta s$ and the values indicated by the last row of Table 1 for the KdV parameters $\varepsilon$, $\mu$ and $A$ are indicated by Table 2.

Table 2

<table>
<thead>
<tr>
<th>$\Delta s$</th>
<th>$x_{f=\text{min}}$ for $f(x) = \text{minimum}$</th>
<th>$\Delta t_{\text{max}}$</th>
<th>$\Delta t_{\text{Vli}}$</th>
<th>$\Delta t_{\text{max}}/\Delta t_{\text{Vli}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.8661</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.15</td>
<td>0.865</td>
<td>0.00654</td>
<td>0.0042</td>
<td>1.537</td>
</tr>
<tr>
<td>0.50</td>
<td>0.8774</td>
<td>0.2219</td>
<td>0.1471</td>
<td>1.509</td>
</tr>
</tbody>
</table>

From Table 2 one finds that the value $x_{f=\text{min}}$ of $x$ ensuring a minimum of the function $f(x, \Delta s)$ is approximately equal to: $x_{f=\text{min}} \approx 0.871 \pm 0.007$. Introducing now this approximate (but rather accurate) value in the expression (10) of the stability condition of the 2 steps discretized FD schemes describing the KdV solitons propagation, one obtains (for the FD schemes avoiding the effects of boundary conditions, etc) a considerably more accurate (than the Vliegenthart’s one) approximation of the stability condition, as:
\[
\Delta t \leq \frac{1.15 \times \Delta s}{[\varepsilon \cdot A - 3\mu/(\Delta s)^2]}. \tag{13}
\]

The accomplished numerical study pointed out that the approximation (12) presents a good accuracy for a broad spectrum of values of parameters \(A, \varepsilon, \mu, \Delta s\), the relative errors affecting the obtained maximum values \(\Delta t_{\text{max}}\) (relative to the exact values from relation (10)) being less than 1%.

### 5. Derivation of the Stability Condition of the 4 Steps Discretized FD Schemes of the KdV Solitons Propagation

Returning to the derivation of the stability condition (10), it is very easy to find that both the well-known Vliegenthart’s stability condition (11) and the more accurate stability condition (10) correspond to the 2 steps discretization (6) of the equation (5) of KdV solitons propagation.

If this 2 steps discretization is substituted by the 4 steps one [1]:

\[
\frac{\partial u}{\partial s} \equiv u_s = \frac{u(s - 2\Delta s) - 8u(s - \Delta s) + 8u(s + \Delta s) - u(s + 2\Delta s)}{12\Delta s}, \tag{14}
\]

and – for a better stability of the FD scheme – the displacement value \(u\) intervening in the nonlinear term of the KdV equation (5) is substituted by its average with the 2 neighbor displacements [15], [12a], p. 385:

\[
u \equiv u^n_j \rightarrow \frac{u^n_{j-1} + u^n_j + u^n_{j+1}}{3},
\]

the expression of the 4 steps discretization of the KdV equation becomes:

\[
u_{j+1}^n = \nu_{j-1}^n - \varepsilon \frac{\Delta t}{18\Delta s} \left( \nu_{j-1}^n + \nu_j^n + \nu_{j+1}^n \right) \cdot \left( \nu_{j-2}^n - 8\nu_{j-1}^n + 8\nu_{j+1}^n - \nu_{j+2}^n \right) - \mu \frac{\Delta t}{(\Delta s)^2} \left( \nu_{j+2}^n - 2\nu_{j+1}^n + 2\nu_{j-1}^n - \nu_{j-2}^n \right). \tag{15}
\]

Introducing in this equation the Fourier expansion (1) of the exact solution \(u_j^n = u(n \cdot \Delta t, j \cdot \Delta s)\) of the pulse propagation equation (5) and using the same derivation procedure of the characteristic equation of the amplification factors \(g_k(k \cdot \Delta s)\) as for the 2 steps discretized KdV equation [see relations (7)-(10)], one obtains the stability condition for the 4 steps discretized KdV equation (5):
Study of the Vliegenthart’s stability condition

\[ \Delta t \leq \frac{\Delta s}{\max \left| 2 \sin \xi \left[ \frac{\varepsilon |u|}{2} + (1 - \cos \xi) \left[ \frac{\varepsilon |u|}{6} - \frac{\mu}{(\Delta s)^2} \right] \right] \right|} = \Delta t_{\text{max}} \]

\[ = \frac{\Delta s}{\max \left| 2x \left[ \frac{\varepsilon A}{2} + (1 + \sqrt{1 - x^2}) \left[ \frac{\varepsilon A}{6} - \frac{\mu}{(\Delta s)^2} \right] \right] \right|} = \Delta t_{\text{max}} \]

The previous condition leads to the evaluation approximation:

\[ \Delta t \leq \frac{0.74 \times \Delta s}{\varepsilon \cdot A - 2\mu / (\Delta s)^2} \] (16)

and to somewhat larger value (1.61) of the ratio \( \Delta t_{\text{max}} / \Delta t_{V} \) for \( A = 2, \varepsilon = 0.1, \mu = 0.2, \Delta s = 0.15 \).

Finally, using the substitution \( u = V_{oo} + \varepsilon \cdot w \) [where \( V_{oo} \) is the pulse velocity into an ideal (linear and non-dispersive) medium] in the more general KdV equation expression:

\[ w_t + V_{oo} w_x + \varepsilon \cdot w \cdot w_x + \mu \cdot w_{xxx} = 0, \] (17)

one obtains a particular expression (\( \varepsilon = 1 \)) of equation (5):

\[ u_t + u \cdot u_x + \mu \cdot u_{xxx} = 0. \] (5’)

The FD scheme (15), with \( \varepsilon = 1 \) (corresponding to the last equation) has the maximum time step:

\[ \Delta t \leq \frac{\Delta s}{\max \left| x \left[ A + \left( 1 + \sqrt{1 - x^2} \right) \left( \frac{A}{3} - 2\mu / (\Delta s)^2 \right) \right] \right|} = \Delta t_{\text{max}}. \] (18)

The accomplished numerical calculations led to the values: \( x_{f_{\text{min}}} \approx 0.855 \) and \( \Delta t_{\text{max}} \approx 0.00705 \) (for the above indicated values of the KdV parameters), with 70% larger than the corresponding Vliegenthart’s limit (11). For this case, the approximation relation is:

\[ \Delta t \leq \frac{0.74 \times \Delta s}{A - 2\mu / (\Delta s)^2}. \] (18’)

5. FINAL REMARKS

As it is well-known, the Courant-Friedrichs-Lewy [16] stability condition of the FD schemes intended to the simulation of the waves propagation require that:

\[ \frac{\Delta s}{\Delta t} \geq V_{\text{wave}}, \]

equivalent to the knowledge of all information about the wave propagation to each step of the FD scheme. Taking into account that the obtained stability conditions for the KdV pulses propagation have the form [see e.g. (10)]:

\[ \frac{\Delta s}{\Delta t} \geq \sin(k \cdot \Delta s) \cdot \left[ \varepsilon \cdot u - 2 \frac{\mu}{(\Delta s)^2} \cdot (1 - \cos(k \cdot \Delta s)) \right], \]

it seems that the modulus from the right part has the meaning of propagation velocity of the simulated FD soliton for the component \( k \cdot \Delta s \).

As it concerns the upper threshold \( \Delta t_{\text{max}} \) of the time step predicted by the stability conditions, we will underline that it determines the minimum value \( N_{\text{min}} \) of the number of iterations which cover a certain time \( t \):

\[ N_{\text{min}} = N(t_{\text{min}}) = \frac{t}{\Delta t_{\text{min}}} < \frac{t}{\Delta t} = N(\Delta t). \]

6. CONCLUSIONS

The accomplished study pointed out both the rather rough approximate (in fact only a sufficient condition, not a necessary one) character of the frequently used Vliegenthart’s stability conditions for the KdV solitons, as well as the rigorous stability conditions corresponding to the FD schemes with two and four steps, respectively, of the Korteweg-de Vries solitons propagation. Besides the exact analytical stability conditions, some numerical calculations were also achieved so that it was possible to obtain suitable approximate expressions in different particular cases.

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REFERENCES


