ON A ONE-DIMENSIONAL NONLINEAR COUPLED SYSTEM OF EQUATIONS IN THE THEORY OF THERMOELASTICITY

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The thermoelasticity deals with predicting the thermomechanical treatment of elastic solids and it is a generalization of the classical theory of elasticity and the theory of thermal conductivity. In this manuscript, the system of nonlinear partial differential equations such as the Cauchy problem which appears in a one-dimensional nonlinear coupled system of equations in the theory of thermoelasticity is studied. The homotopy analysis method was used to perform successfully the numerical calculations.

\textbf{Key words:} Approximate solution; Thermo-elasticity; Cauchy problem; Nonlinear equations.


1. INTRODUCTION

The thermoelasticity explains a vast domain of phenomena, the theory of thermoelasticity is concerned with predicting the thermomechanical treatment of elastic solids. It is the extension of both the classical theory of elasticity and the theory of thermal conductivity. Duhamel [1] presented the modern theory of thermoelasticity in 1837. However, thermoelasticity was one of the first domains in coupled field theory that attracted the attention of mathematicians; the interest to the models characterizing thermomechanical coupling has been revitalized by novel significant practical problems, including those that are at the cutting edge of the current technological innovations. Despite over a century of research on thermoelasticity, many problems of current interest are intractable when solutions are designed by classical

methods. This fact has led some investigators to discuss numerical ways for solving thermoelasticity problems.

Many methods for solving the thermoelasticity equations have been extended, scientific researchers have solved some problems concerning the propagation of thermoelastic waves. Oden et al. [2] employed boundary element method for thermoelastic problems and the similar method was used for analysing the thermal and mechanical shock in a two-dimensional (2D) domain in [3]. Nickell and Sackman utilized the extended Ritz method for coupled thermoelastic boundary-value problems, see Ref. [4]. Using the Green-Lindsay theory, Agarwal [5] investigated both thermoelastic and magneto-thermoelastic plane wave propagation in an infinite non-rotation medium. Recently, Jafarian et al. [6] used homotopy analysis method (HAM) to study the coupled nonlinear magneto-thermoelasticity equations under influence of rotation. It is the purpose of this article to develop an efficient approach, namely the HAM, to the solution of coupled problems of thermoelasticity, which should be easy to be generalized for more unintelligible models. The HAM was examined by Jafarian et al. [7] for the approximate solution of Kadomtsev-Petviashvili-II equation. Recently, this method has been successfully employed to solve many kinds of problems in science and engineering [8–18]. Unlike all other analytic methods, HAM offers a simple way to adjust and control the convergence region of solution series by choosing suitable values for auxiliary parameter $h$. The validity and reliability of this method is independent of whether or not there exist small parameters in the considered problem. So, no matter governing equations and boundary/initial conditions contain small or large quantities or not, this method can be employed.

This manuscript is organized as follows. The model is presented Section 2. In Section 3, we extend the application of HAM in order to construct approximate solutions of the Cauchy problem arising in one dimensional nonlinear thermoelasticity. In Section 4, the results are employed to illustrate the convergence, the accuracy and the computational efficiency of this approach. In Section 5, the convergence of the HAM series solution is analysed. Finally, conclusions are presented in Section 6.

2. THE MODEL

The aim of this paper is to solve the following system of nonlinear coupled one-dimensional partial differential equations of thermoelasticity as presented in [19–21].

\[ u_{tt} - a(u_x, \theta)u_{xx} - b(u_x, \theta)\theta_x = f(x, t), \]
\[ c(u_x, \theta)\theta_t + b(u_x, \theta)u_{xt} - d(\theta)\theta_{xx} = g(x, t), \]

under the following initial conditions

\[ u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad \theta(x, 0) = \theta^0(x), \]
where \( u(x,t) \) is the body displacement form equilibrium and \( \theta(x,t) \) is the difference of the body’s temperature from a reference \( T_0 = 0 \), subscripts denote partial derivatives and \( a, b, c \) and \( d \) are given smooth functions. In order to illustrate the effectiveness of the method, a model is used. Let us define \( a, b, c, d, u^0, u^1 \) and \( \theta^0 \) by [21]

\[
\begin{align*}
  a(u_x, \theta) &= 2 - u_x \theta, \quad b(u_x, \theta) = 2 + u_x \theta, \\
  c(u_x, \theta) &= 1, \quad d(\theta) = \theta,
\end{align*}
\]

\( u^0(x) = \frac{1}{1 + x^2}, \quad u^1(x) = 0, \quad \theta^0(x) = \frac{1}{1 + x^2}. \)

We replace the right-hand side of the above equations by

\[
\begin{align*}
  f &= \frac{2}{1 + x^2} - \frac{2(1 + t^2)(3x^2 - 1)}{(1 + x^2)^3} a(w, v) - \frac{2x(1 + t)}{(1 + x^2)^2} b(w, v), \\
  g &= \frac{4xt}{1 + x^2} c(w, v) - \frac{2(3x^2 - 1)(1 + t)}{(1 + x^2)^3} d(v),
\end{align*}
\]

where \( a, b, c \) and \( d \) are defined above and

\[
\begin{align*}
  w &= w(x, t) = -\frac{2x(1 + t^2)}{(1 + x^2)^2}, \quad v = v(x, t) = \frac{1 + t}{1 + x^2}.
\end{align*}
\]

Thus, the exact solution \( u(x,t) \) and \( \theta(x,t) \) of the system (1)-(2) is

\[
\begin{align*}
  u(x, t) &= \frac{1 + t^2}{1 + x^2}, \quad v(x, t) = \frac{1 + t}{1 + x^2}.
\end{align*}
\]

3. THE METHOD

To solve the following by HAM [22], according to the initial conditions denoted in equations (3), we have

\[
\begin{align*}
  u_{tt} - 2u_{xx} + u_x u_{xx} \theta + 2\theta_x + u_x \theta_x \theta - f(x, t) &= 0, \\
  \theta_t + 2u_{xt} + u_x u_{xt} \theta - \theta \theta_{xx} - g(x, t) &= 0.
\end{align*}
\]

It is natural to choose

\[
\begin{align*}
  u_0(x, t) &= \frac{1}{1 + x^2}, \quad \theta_0(x, t) = \frac{1}{1 + x^2}
\end{align*}
\]

and the linear operators

\[
\begin{align*}
  L_1[\varphi_1(x,t; q)] &= \frac{\partial^2 \varphi_1}{\partial t^2}, \quad L_2[\varphi_2(x,t; q)] = \frac{\partial \varphi_2}{\partial t},
\end{align*}
\]

with property \( L_1[c_1 + tc_2] = 0, \quad L_2[c_3] = 0 \), where \( c_1, c_2 \) and \( c_3 \) are constant coefficients. The nonlinear operators \( N_1 \) and \( N_2 \) are defined as
\[
N_1[\phi_1, \phi_2] = \frac{\partial^2 \phi_1}{\partial t^2} - 2 \frac{\partial^2 \phi_1}{\partial x \partial t} + \frac{\partial \phi_1}{\partial x} \frac{\partial^2 \phi_1}{\partial x^2} + 2 \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_2}{\partial x} - f(x, t) = 0,
\]
\[
N_2[\phi_1, \phi_2] = \frac{\partial \phi_2}{\partial t} + 2 \frac{\partial^2 \phi_1}{\partial t \partial x} + \frac{\partial \phi_1}{\partial x} \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial \phi_2}{\partial x} \frac{\partial^2 \phi_2}{\partial x^2} - g(x, t) = 0.
\]

Using the above definitions, with assumption \(H_1(x, t) = H_2(x, t) = 1\), we develop the zero-th order deformation equations as
\[
(1-q)L_1[\phi_1(x, t; q) - u_0(x, t)] = qh_1N_1[\phi_1, \phi_2],
\]
\[
(1-q)L_2[\phi_2(x, t; q) - \theta_0(x, t)] = qh_2N_2[\phi_1, \phi_2].
\]

Obviously, when \(q=0\) and \(q=1\)
\[
\phi_1(x, t; 0) = u_0(x, t), \quad \phi_1(x, t; 1) = u(x, t),
\]
\[
\phi_2(x, t; 0) = \theta_0(x, t), \quad \phi_2(x, t; 1) = \theta(x, t).
\]

As a result, we obtain the \(m\)-th order deformation equations
\[
L_1[u_m(x, t) - \chi_m u_{m-1}(x, t)] = h_1R_{1m}(\tilde{u}_{m-1}, \tilde{\theta}_{m-1}), \quad (6)
\]
\[
L_2[\theta_m(x, t) - \chi_m \theta_{m-1}(x, t)] = h_2R_{2m}(\tilde{u}_{m-1}, \tilde{\theta}_{m-1}), \quad (7)
\]

where
\[
R_{1m}(\tilde{u}_{m-1}, \tilde{\theta}_{m-1}) = \frac{\partial^2 u_{m-1}}{\partial t^2} - 2 \frac{\partial^2 u_{m-1}}{\partial x \partial t} + \sum_{n=0}^{m-1} \frac{\partial^2 u_{m-1-n}}{\partial x^2} \left[ \sum_{j=0}^{n} \frac{\partial u_j}{\partial x} \theta_{n-j} \right] - (1 - \chi_m)f(x, t) = 0,
\]
\[
R_{2m}(\tilde{u}_{m-1}, \tilde{\theta}_{m-1}) = \frac{\partial \theta_{m-1}}{\partial t} + 2 \frac{\partial^2 u_{m-1}}{\partial t \partial x} + \sum_{n=0}^{m-1} \frac{\partial^2 u_{m-1-n}}{\partial x \partial t} \left[ \sum_{j=0}^{n} \frac{\partial u_j}{\partial x} \theta_{n-j} \right] - \sum_{j=0}^{m-1} \theta_j \frac{\partial \theta_{m-1-j}}{\partial x^2} - (1 - \chi_m)g(x, t) = 0.
\]

Now, the solution of the \(m\)-th order deformation equations (6)-(7) for \(m \geq 1\) becomes
\[
u_m(x, t) = \chi_m u_{m-1}(x, t) + h_1L_{1m}^{-1}[R_{1m}(\tilde{u}_{m-1}, \tilde{\theta}_{m-1})],
\]
\[
\theta_m(x, t) = \chi_m \theta_{m-1}(x, t) + h_2L_{2m}^{-1}[R_{2m}(\tilde{u}_{m-1}, \tilde{\theta}_{m-1})].
\]
Finally, we have

\[ u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t), \]

\[ \theta(x, t) = \theta_0(x, t) + \sum_{m=1}^{\infty} \theta_m(x, t). \]

Fig. 1 – The three-dimensional plot of the 4th-order approximate solutions: A: \( u(x, t) \), B: \( \theta(x, t) \).

4. RESULTS

In this section we carry out the numerical simulation to illustrate the flexibility of the numerical method. In order to illustrate the effectiveness of the method, the equations (1)-(2) are used. The three-dimensional surfaces plotted in Figure 1 are the 4-th order approximate solutions. Tables 1 and 2 show the absolute errors for differences between the exact solutions and the 4-th order approximate solutions obtained by HAM at some points. Besides, the behaviour of the exact and approximate solutions are illustrated in Figures 2 and 3. Comparison of the result obtained by HAM with exact solution reveals that the accuracy of the new method. It is evident that the overall errors can be made smaller by adding new terms from the iteration formulas. The result show that our method was capable of solving the nonlinear problems.
5. CONVERGENCE OF THE ANALYTIC SOLUTION

The solutions given by the HAM contain an auxiliary parameter $h$, which can be applied to control and adjust the convergence region and rate of the HAM solution series. It is interesting that the convergence rate of the approximation series depends
Fig. 4 – The $h$-curves given by the 4th-order approximate solution of the equations (4)-(5).

Table 1

<table>
<thead>
<tr>
<th>$x$</th>
<th>$t = 0.25$</th>
<th>$t = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5.1708 E-07</td>
<td>1.8687 E-05</td>
</tr>
<tr>
<td>6</td>
<td>2.0975 E-07</td>
<td>7.4807 E-06</td>
</tr>
<tr>
<td>7</td>
<td>9.7368 E-08</td>
<td>3.4337 E-06</td>
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<tr>
<td>8</td>
<td>4.9976 E-08</td>
<td>1.7458 E-06</td>
</tr>
<tr>
<td>9</td>
<td>2.7718 E-08</td>
<td>9.6063 E-07</td>
</tr>
<tr>
<td>10</td>
<td>1.6348 E-08</td>
<td>5.6281 E-07</td>
</tr>
<tr>
<td>11</td>
<td>1.0136 E-08</td>
<td>3.4697 E-07</td>
</tr>
<tr>
<td>12</td>
<td>6.5503 E-09</td>
<td>2.2312 E-07</td>
</tr>
<tr>
<td>13</td>
<td>4.3829 E-09</td>
<td>1.4866 E-07</td>
</tr>
<tr>
<td>14</td>
<td>3.0211 E-09</td>
<td>1.0209 E-07</td>
</tr>
<tr>
<td>15</td>
<td>2.1365 E-09</td>
<td>7.1959 E-08</td>
</tr>
</tbody>
</table>

upon the value auxiliary parameter $h$. As pointed by Liao [22], the valid region of $h$ is a line segment nearly parallel to the horizontal axis. In general, by means of the so-called $h$-curve, it is straightforward to choose an appropriate range for $h$ which ensures the convergence of the solution series. To study the influence of $h$ on the convergence of solution, the $h$-curves of $u(5,0.25)$ and $\theta(5,0.25)$ are plotted for selected values of constant numbers, as shown in Figure 4. For better presentation, these valid regions have been listed in table 3.

6. CONCLUSIONS

In this manuscript, HAM has been effectively exerted to discover the approximate solution of the Cauchy problem arising in one-dimensional nonlinear thermoe-
Table 2

Absolute errors for the 4-th order approximate solution of $\theta(x,t)$ given by HAM ($h = -1$).

<table>
<thead>
<tr>
<th>$x$</th>
<th>$t = 0.25$</th>
<th>$t = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.8297 E-05</td>
<td>2.9986 E-04</td>
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<td>6</td>
<td>7.5594 E-06</td>
<td>1.2353 E-04</td>
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<tr>
<td>7</td>
<td>3.5571 E-06</td>
<td>5.7979 E-05</td>
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<td>8</td>
<td>1.8449 E-06</td>
<td>3.0008 E-05</td>
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<tr>
<td>9</td>
<td>1.0317 E-06</td>
<td>1.6752 E-05</td>
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<td>10</td>
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<td>6.1855 E-06</td>
</tr>
<tr>
<td>12</td>
<td>2.4799 E-07</td>
<td>4.0119 E-06</td>
</tr>
<tr>
<td>13</td>
<td>1.6660 E-07</td>
<td>2.6929 E-06</td>
</tr>
<tr>
<td>14</td>
<td>1.1525 E-07</td>
<td>1.8613 E-06</td>
</tr>
<tr>
<td>15</td>
<td>8.1739 E-08</td>
<td>1.3194 E-06</td>
</tr>
</tbody>
</table>

Table 3

The admissible values of $h$ derived from Fig. 4.

<table>
<thead>
<tr>
<th>$u(x,t)$</th>
<th>$-1.15 \leq h_1 \leq -0.90$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta(x,t)$</td>
<td>$-1.05 \leq h_2 \leq -0.95$</td>
</tr>
</tbody>
</table>

elasticity. We reported that the HAM was a very efficient and a powerful technique in finding the solutions of the proposed equations. A major difference in analysis between this method and others is that HAM can be used as an appropriate approach for controlling the convergence of approximation series. The results also indicate that the obtained approximate solutions corresponded with the exact solution. The method was applied in a straightforward way without using the linearisation or perturbation. Moreover, the approximate solutions obtained by this method have high convergence speed and the drawn diagrams confirm this issue.

REFERENCES