A NEW METHOD TO CONSTRUCT TRAVELLING WAVE SOLUTIONS FOR THE KLEIN-GORDON-ZAKHAROV EQUATIONS

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In this paper, we construct the travelling wave solutions to the Klein-Gordon-Zakharov equations (KGZEs) by a new method. Based on this method, we obtain abundant exact travelling wave solutions of KGZEs with arbitrary parameters. The travelling wave solutions are expressed by the hyperbolic functions, trigonometric functions and rational functions.

Key words: Klein-Gordon-Zakharov equations (KGZEs), travelling wave solutions, a new method.

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1. INTRODUCTION

During the last decades, the investigation of the exact travelling wave solutions to nonlinear partial differential equations(NLPDEs) plays an important role in the study of complex physical and mechanic phenomena. Lots of effective methods for obtaining exact solutions of NLPDEs have been reported, such as the trigonometric function series method [1, 2], the modified mapping method and the extended mapping method [3], the modified trigonometric function series method [4], the dynamical system approach and the bifurcation method [5, 6], the infinite series method and Jacobi elliptic function expansion method [7], the exp-function method [8], the multiple exp-function method[9], the transformed rational function method [10], the symmetry algebra method (consisting of Lie point symmetries) [11], the Wronskian technique [12], the linear superposition principle [13] and so on.

Recently, Wang *et al.* [14] proposed the $(\frac{G'}{G})$-expansion method to construct the travelling wave solutions for NLPDEs. The method is based on the homogeneous balance principle and linear ordinary differential equation(LODE) theory. It is assume that the travelling wave solutions can be expressed by a polynomial in $(\frac{G'}{G})$, and that G'' satisfies a seconder order LODE $G'' + \lambda G' + \mu G = 0$. The degree of the polynomial can be determined by the homogeneous balance between the highest order derivative and linear terms appearing in the given NLPDEs. The coefficients of the polynomial can be obtained by solving a set of algebraic equations. Recently, the $(\frac{G'}{G})$-expansion method has been successfully applied to obtain exact solutions for a variety of NLPDEs [15-19]. In particular, Miao and Zhang [20] proposed a new method called the modified $(\frac{G'}{G})$-expansion method to construct travelling wave solutions of the perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity

$$iu_t + u_{xx} + \alpha|u|^2u + i[\gamma_1 u_{xxx} + \gamma_2|u|^2u_x + \gamma_3(|u|^2)_x u] = 0. \tag{1}$$

In their contribution, they obtained new travelling wave solutions of Eq.(1) by using the modified $(\frac{G'}{G})$-expansion method and G'' satisfies a seconder order LODE $G'' + \mu G = 0$. In fact the $(\frac{G'}{G})$-expansion method is just a variant of the transformation method which transforms nonlinear partial differential equations into integrable ordinary differential equations to solve, see Ref.[2].

In this paper, we investigate the Klein-Gordon-Zakharov [21] or [22]

$$u_{tt} - u_{xx} + u + \alpha nu = 0, \tag{2}$$

$$n_{tt} - n_{xx} = \beta(|u|^2)_{xx}, \tag{3}$$

where function $u(x,t)$ denotes the fast time scale component of electric field raised by electrons and the function $u(x,t)$ denotes the deviation of ion density from its equilibrium. Here $u(x,t)$ is a complex function, $n(x,t)$ is a real function, α, β are two nonzero real parameters. This system describes the interaction of the Langmuir wave and the ion acoustic wave in a high frequency plasma or the propagation of strong turbulence of the Langmuir wave in a high frequency plasma. More details are presented in Ref.[21] and [22].

Recently, applying trigonometric function series method, Zhang[1] studied the new exact travelling wave solutions of the Klein-Gordon equation(KGE)

$$u_{tt} - u_{xx} + \alpha u - \beta u^3 = 0. \tag{4}$$

Eq.(4) describes the propagation of dislocations within crystals, the Bloch wall motion of magnetic crystals, the propagation of a "splay wave" along a lied membrane, the unitary theory for elementary particles and the propagation of magnetic flux on a Josephson line, etc. More details are presented in Ref.[1] or [22]. Quite recently, Xiao and Zhang [23] studied the travelling wave solutions of Eq.(4) by using $(\frac{G'}{G})$-expansion method and showed that the travelling wave solutions are expressed in terms of hyperbolic functions, the trigonometric functions and the rational functions. More recently, many exact solutions for Zakharov equations have been successfully obtained by using the extended tanh-expansion method, the extended hyperbolic function method, the F-expansion method[24-27]. It is worth mentioning that Shang er al.[21] obtained the multiple exact explicit solutions of the KGZEs (2) and (3). These solutions include that of the solitary wave solutions of bell-type for u and n, the solitary wave solutions of kink-type for u and bell-type for n, the solitary wave solutions of a compound of the bell-type and the kink-type for u and

n, the singular travelling wave solutions, the periodic travelling wave solutions of triangle functions type, and solitary wave solutions of rational function type. Quite recently, Zhang *et al.* [22] investigated the new exact travelling wave solutions for the KGZEs (2) and (3) by using the modified trigonometric function series method benefited to the ideas of Zhang *et al.*[4]. These travelling wave solutions are expressed by the hyperbolic functions and the rational functions. Ismail and Biswas [29] investigated the one-soliton solution of the KGZEs (2) and (3) with power law nonlinearity by using the solitary wave ansatz method. The solutions are obtained both in (1 + 1) and (1 + 2) dimensions. For related papers and Solitons in the other physical model, including Klein-Gordon equation,the Bretherton equation,Ginzburg Landau equation, Benney-Luke equation, DWDM systems, nonlinear Schrödinger's equation, Boussinesq-Burgers equation, see Ref.[31-43]. For higher dimensional KGZEs and $\alpha = 1$, $\beta = 1$, by using the dynamical systems approach, Li [30] considered the existence of exact explicit bounded travelling wave solutions of following equations:

$$\phi_{tt} - \triangle\phi + \phi + \phi\psi = 0,\ \psi_{tt} - c^2\triangle\psi = \Delta|\phi|^2,$$

where $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian operator, $x \in R^n$, c is the propagation speed of a wave. More details are presented in Ref.[15].

However, in our novel contribution, we will propose a new method, which can be thought of as an extension of the $(\frac{G'}{G})$-expansion method. The key idea of this method is that the travelling wave solutions of NLPDEs can be expressed by a polynomial in two variables $(\frac{G'}{G})$ and $(\frac{1}{G})$, in which $G = G(\xi)$ satisfies a seconder order LODE. The degree of the polynomial can be determined by the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given NLPDEs, and the coefficients of the polynomial can be obtained by solving a set of algebraic equations. More details are present in section 2.

2. DESCRIPTION OF THE NEW METHOD

In this section, we shall describe the main idea of our present method for constructing travelling wave solutions of NLPDEs.

Assume that a seconder order LODE

$$G'' + \lambda G = \mu, \tag{5}$$

where

$$\phi = \frac{G'}{G},\ \psi = \frac{1}{G}. \tag{6}$$

It follows from (5) and (6) that

$$\phi' = -\phi^2 + \mu\psi - \lambda,\ \psi' = -\phi\psi. \tag{7}$$

To facilitate further on our analysis, we discuss the general solutions of the LODE (5) as follows:

Case 1. When $\lambda < 0$, the general solutions of the LODE (2.1)

$$G(\xi) = A_1 \sinh\sqrt{-\lambda}\xi + A_2 \cosh\sqrt{-\lambda}\xi + \frac{\mu}{\lambda}, \tag{8}$$

and we get

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}(\phi^2 - 2\mu\psi + \lambda),$$

where A_1 and A_2 are two constants and $\sigma = A_1^2 - A_2^2$.

Case 2. When $\lambda > 0$, the general solutions of the LODE (5)

$$G(\xi) = A_1 \sinh\sqrt{\lambda}\xi + A_2 \cosh\sqrt{\lambda}\xi + \frac{\mu}{\lambda}, \tag{9}$$

and we get

$$\psi^2 = \frac{\lambda}{\lambda^2\rho - \mu^2}(\phi^2 - 2\mu\psi + \lambda),$$

where A_1 and A_2 are two constants and $\rho = A_1^2 + A_2^2$.

Case 3. When $\lambda = 0$, the general solutions of the LODE (5)

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{10}$$

and we get

$$\psi^2 = \frac{\lambda}{A_1^2 - 2\mu A_2}(\phi^2 - 2\mu\psi),$$

where A_1 and A_2 are two constants.

Suppose that a NLPDE is given by

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{11}$$

where $u = u(x,t)$ is an unknown function and F is a polynomial. Now, we are position to show the main steps of the new method.

Step 1. To construct the travelling wave solutions of (11), we introduce the wave transformation

$$u(x,t) = u(\xi),\ \xi = x - \eta t + \xi_1, \tag{12}$$

where ξ_1, η are constants. Substituting (12) into (11), we obtain the following ODE

$$P(u, u', u'', u''', \ldots) = 0, \tag{13}$$

Step 2. Assume that the solution of Eq.(13) can be expressed by a polynomial in ϕ and ψ as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{j=1}^{N} b_j\phi^{j-1}\psi, \tag{14}$$

where $G = G(\xi)$ satisfies the LODE (2.1), $a_i (i = 0, 1, \ldots, N)$, $b_j (j = 1, \ldots, N)$, λ, μ are constants to be determined later, and the positive integer N can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in ODE (13).

Step 3. Substituting the solution (14) together with (2.8) into (13) yields an algebraic equations including powers of $\phi^i \psi^j$. Equating the coefficients of each power of $\phi^i \psi^j$ to zero gives a system of algebraic equations for a_i, b_j, η λ, μ, A_1 and A_2.

Step 4. Solve the algebraic equations in the step 3 with the aid of `Mathematica`. Then substituting the values of parameters, one can obtain the travelling wave solutions of (11).

3. TRAVELLING WAVE SOLUTIONS OF KGZEs (2) AND (3)

In this section, we shall illustrate our present new method in detail by constructing the travelling wave solutions of travelling wave solutions of KGZEs (2) and (3).

Assume that Eq.(2) has travelling wave solutions in the form [21] or [22]

$$u(x,t) = \Phi(x,t) \exp(i(kx + \omega t + \xi_0)), \tag{15}$$

where $u(x,t)$ is a real-valued function, k, ω are two real constants to be determined, ξ_0 is an arbitrary constant. Substituting (15) into (2)-(3) yields

$$\Phi_{tt} - \Phi_{xx} + (k^2 - \omega^2 + 1)\Phi + \alpha n \Phi = 0, \tag{16}$$

$$\omega \Phi_t - k \Phi_x = 0, \tag{17}$$

$$n_{tt} - n_{xx} = \beta (\Phi^2)_{xx}. \tag{18}$$

By virtue of (17), we assume

$$\Phi(x,t) = \Phi(\xi) = \Phi(\omega x + kt + \xi_1), \tag{19}$$

where ξ_1 is an arbitrary constant. Substituting (19) into (16), we have

$$n(x,t) = \frac{(\omega^2 - k^2)\Phi''(\xi)}{\alpha \Phi(\xi)} + \frac{(\omega^2 - k^2 - 1)}{\alpha}. \tag{20}$$

Hence, we can also assume

$$n(x,t) = \Psi(\xi) = \Psi(\omega x + kt + \xi_1). \tag{21}$$

Substituting (21) into (18) and integrating the resultant equation twice with respect to ξ, we obtain

$$\Psi(\xi) = \frac{\beta \omega^2 \Phi^2(\xi)}{k^2 - \omega^2} + C, \tag{22}$$

where C is an integration constant. Without loss of generality, we assume $C=0$. It follows from (16) and (22) that

$$\Phi''(\xi)+\frac{k^2-\omega^2+1+\alpha C}{k^2-\omega^2}\Phi(\xi)+\frac{\alpha\beta\omega^2}{(k^2-\omega^2)^2}\Phi^3(\xi)=0. \tag{23}$$

For simplicity, we assume $A=\frac{k^2-\omega^2+1+\alpha C}{k^2-\omega^2}$, $B=\frac{\alpha\beta\omega^2}{(k^2-\omega^2)^2}$, thus (23) leads to ordinary differential equation (ODE)

$$\Phi''(\xi)+A\Phi(\xi)+B\Phi^3(\xi)=0. \tag{24}$$

In what follows, we will discuss the travelling wave solutions to (24).

By balancing the highest order derivative term Φ'' and the nonlinear term Φ^3 in (24), we obtain $N=1$ in (14). So, we assume that (24) has solution in the form

$$\Phi(\xi)=a_1\phi+b_1\psi, \tag{25}$$

where a_1 and b_1 are constants to be determined later and satisfy $a_1^2+b_1^2\neq 0$. Next, there are three cases to be investigated and we give the corresponding travelling wave solutions.

Case 1. When $\lambda<0$. Substituting (25) into (24), the left-hand side of (24) becomes a polynomial in ϕ and ψ. Setting its coefficients to zero yields a system of algebraic equations as follows:

$$\begin{aligned}
\phi^3:&\ 2a_1+B(a_1^3-\frac{3a_1b_1^2\lambda}{\lambda^2\sigma+\mu^2})=0,\\
\phi^2\psi:&\ 2b_1+B(3a_1^2b_1-\frac{b_1^3\lambda}{\lambda^2\sigma+\mu^2})=0,\\
\phi^2:&\ b_1\mu\lambda-\frac{2Bb_1^3\mu\lambda^2}{\lambda^2\sigma+\mu^2}=0,\\
\phi\psi:&\ -a_1\mu+\frac{2Ba_1^2b_1\mu\lambda}{\lambda^2\sigma+\mu^2}=0,\\
\phi:&\ 2a_1\lambda+Aa_1-\frac{3Ba_1b_1^2\lambda^2}{\lambda^2\sigma+\mu^2}=0,\\
\psi:&\ b_1\lambda(\lambda^2\sigma-\mu^2)+Ab_1(\lambda^2\sigma+\mu^2)+\frac{Bb_1^3\lambda^2(3\mu^2-\lambda^2\sigma)}{\lambda^2\sigma+\mu^2}=0,\\
\psi^0:&\ b_1\mu\lambda^2-\frac{2Bb_1^3\mu\lambda^2}{(\lambda^2\sigma+\mu^2)^2}=0.
\end{aligned}$$

Solving the above system by `Mathematica`, we have

Case(i). If $A<0$ and $B>0$ or (<0), then

$$a_1=0,\ b_1=\pm\sqrt{\frac{2A\sigma}{B}},\sigma>0 \text{ or } <0,\lambda=A,\mu=0.$$

Case(ii). If $A>0$ and $B<0$, then

$$a_1=\pm\sqrt{\frac{-2}{B}},\ b_1=0,\sigma=\text{an arbitrary constant},\lambda=-\frac{A}{2},\mu=0.$$

Case(iii). If $A>0$ and $B<0$, then

$$a_1=\pm\sqrt{\frac{-1}{2B}},\ b_1=\pm\sqrt{-\frac{4A\sigma+\mu^2}{4AB}},\sigma\geq-\frac{\mu^2}{4A^2},\lambda=-2A,\mu=\text{an arbitrary constant}.$$

From the above cases, we obtain the hyperbolic function solutions of (2) and (3) as follows:

Case 1.1. From (15), (25) in **Case(i)**, we get

$$\Phi=b_1\psi=b_1\frac{1}{G}$$

and

$$u_1=\pm\sqrt{\frac{2A(A_1^2-A_2^2)}{B}}\frac{\exp(i(kx+\omega t+\xi_0))}{A_1\sinh\sqrt{-A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{-A}(x-\eta t+\xi_1)} \tag{26}$$

It follows from (21)(22) and (26) that

$$n_1=\frac{\beta\omega^2\Phi^2(\xi)}{k^2-\omega^2}=\frac{\beta\omega^2}{k^2-\omega^2}\frac{2A(A_1^2-A_2^2)}{B}\frac{1}{(A_1\sinh\sqrt{-A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{-A}(x-\eta t+\xi_1))^2}. \tag{27}$$

Case 1.2. From (15), (25) in **Case(ii)**, we get

$$\Phi=a_1\phi=a_1\frac{G'}{G}$$

and

$$u_2=\pm\sqrt{-\frac{2}{B}}\frac{A_1\cosh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)+A_2\sinh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)+A_2\cosh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)}\mathrm{e}^{i(kx+\omega t+\xi_0)} \tag{28}$$

It follows from (21), (22) and (28) that

$$\begin{aligned}n_2&=\frac{\beta\omega^2\Phi^2(\xi)}{k^2-\omega^2}\\&=\frac{\beta\omega^2}{k^2-\omega^2}\frac{2}{B}\left(\frac{A_1\cosh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)+A_2\sinh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)+A_2\cosh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)}\right)^2.\end{aligned} \tag{29}$$

Case 1.3. From (15), (25) in **Case(iii)**, we get

$$\Phi=a_1\phi+b_1\psi=a_1\frac{G'}{G}+b_1\frac{1}{G}$$

and

$$\begin{aligned} u_3 &= \pm\sqrt{-\tfrac{1}{2B}}\,\frac{A_1\cosh\sqrt{2A}(x-\eta t+\xi_1)+A_2\sinh\sqrt{2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{2A}(x-\eta t+\xi_1)}\,\mathrm{e}^{i(kx+\omega t+\xi_0)} \\ &\pm\sqrt{-\tfrac{4A(A_1^2-A_2^2)+\mu^2}{4AB}}\,\frac{\mathrm{e}^{i(kx+\omega t+\xi_0)}}{A_1\sinh\sqrt{2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{2A}(x-\eta t+\xi_1)} = \pm\sqrt{-\tfrac{1}{2B}} \\ &\times\frac{\sqrt{\frac{4A(A_1^2-A_2^2)+\mu^2}{2A}}+A_1\cosh\sqrt{2A}(x-\eta t+\xi_1)+A_2\sinh\sqrt{2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{2A}(x-\eta t+\xi_1)}\,\mathrm{e}^{i(kx+\omega t+\xi_0)} \end{aligned} \tag{30}$$

It follows from (21), (22) and (30) that

$$\begin{aligned} n_3 &= \frac{\beta\omega^2\Phi^2(\xi)}{k^2-\omega^2} = -\frac{\beta\omega^2}{k^2-\omega^2}\frac{1}{2B} \\ &\times\left(\frac{\sqrt{\frac{4A(A_1^2-A_2^2)+\mu^2}{2A}}+A_1\cosh\sqrt{2A}(x-\eta t+\xi_1)+A_2\sinh\sqrt{2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{2A}(x-\eta t+\xi_1)}\right)^2. \end{aligned} \tag{31}$$

Remark 3.1. Taking $A_1 = 0$ and $A_2 > 0$, or $A_1 > 0$ and $A_2 = 0$ respectively, (26) and (27) become

$$\begin{aligned} u_4 &= \pm\sqrt{\frac{-2AA_2^2}{B}}\,\operatorname{sech}\sqrt{-A}(x-\eta t+\xi_1)\exp(i(kx+\omega t+\xi_0)) \\ n_4 &= -\frac{\beta\omega^2}{k^2-\omega^2}\frac{2AA_2^2}{B}\operatorname{sech}^2\sqrt{-A}(x-\eta t+\xi_1) \end{aligned} \tag{32}$$

or

$$\begin{aligned} u_5 &= \pm\sqrt{\frac{-2AA_1^2}{B}}\,\operatorname{csch}\sqrt{-A}(x-\eta t+\xi_1)\exp(i(kx+\omega t+\xi_0)) \\ n_5 &= \frac{\beta\omega^2}{k^2-\omega^2}\frac{2AA_1^2)}{B}\operatorname{csch}^2\sqrt{-A}(x-\eta t+\xi_1). \end{aligned} \tag{33}$$

If $-\frac{\beta\omega^2}{k^2-\omega^2}\frac{2AA_2^2}{B} > 0$, then $n_4 > 0$, and if $-\frac{\beta\omega^2}{k^2-\omega^2}\frac{2AA_2^2}{B} < 0$, then $n_4 < 0$. The solution n_4 is called the Langmuir whistler soliton or the Langmuir pit soliton when $n_4 > 0$ or $n_4 < 0$ respectively.

Remark 3.2. Taking $A_1 = 0$ and $A_2 \neq 0$, or $A_1 \neq 0$ and $A_2 = 0$ respectively, (28) and (29) become

$$\begin{aligned} u_6 &= \pm\sqrt{-\frac{2}{B}}\tanh\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)\exp(i(kx+\omega t+\xi_0)) \\ n_6 &= -\frac{\beta\omega^2}{k^2-\omega^2}\frac{2}{B}\tanh^2\sqrt{\frac{A}{2}}(x-\eta t+\xi_1) \end{aligned} \tag{34}$$

or

$$\begin{aligned} u_7 &= \pm\sqrt{\frac{-2}{B}}\coth\sqrt{\frac{A}{2}}(x-\eta t+\xi_1)\exp(i(kx+\omega t+\xi_0)) \\ n_7 &= -\frac{\beta\omega^2}{k^2-\omega^2}\frac{2}{B}\coth^2\sqrt{\frac{A}{2}}(x-\eta t+\xi_1). \end{aligned} \tag{35}$$

Case 2. When $\lambda > 0$. Similar to the **Case 1**, after solving the system of algebraic equations, we obtain

Case(iv). If $A > 0$ and $B < 0$, then

$$a_1 = 0,\ b_1 = \pm\sqrt{\frac{-2A(A_1^2 + A_2^2)}{B}},\ \lambda = A, \mu = 0.$$

Case(v). If $A < 0$ and $B < 0$, then

$$a_1 = \pm\sqrt{\frac{-2}{B}},\ b_1 = 0,\ \lambda = -\frac{A}{2}, \mu = 0.$$

Case(vi). If $A < 0$ and $B < 0$, then

$$a_1 = \pm\sqrt{\frac{-1}{2B}},\ b_1 = \pm\sqrt{\frac{4A(A_1^2 + A_2^2) - \mu^2}{4AB}},\ A_1^2 + A_2^2 \geq \frac{\mu^2}{4A},$$

$$\lambda = -2A, \mu = \text{arbitrary constant}.$$

From the above cases, we obtain the trigonometric function solutions of (2) and (3) as follows:

Case 2.1. From (15)(25) and **Case(iv)**, we get

$$\Phi = b_1\psi = b_1\frac{1}{G}$$

and

$$u_8 = \pm\sqrt{\frac{-2A(A_1^2 + A_2^2)}{B}}\frac{\exp(i(kx + \omega t + \xi_0))}{A_1\sinh\sqrt{A}(x - \eta t + \xi_1) + A_2\cosh\sqrt{A}(x - \eta t + \xi_1)} \quad (36)$$

It follows from (21)(22) and (36) that

$$n_8 = \frac{\beta\omega^2\Phi^2(\xi)}{k^2 - \omega^2} = -\frac{\beta\omega^2}{k^2 - \omega^2}\frac{2A(A_1^2 + A_2^2)}{B(A_1\sinh\sqrt{A}(x - \eta t + \xi_1) + A_2\cosh\sqrt{A}(x - \eta t + \xi_1))^2}. \quad (37)$$

Case 2.2. From (15)(25) and **Case(v)**, we get

$$\Phi = a_1\phi = a_1\frac{G'}{G}$$

and

$$u_9 = \pm\sqrt{-\frac{2}{B}}\frac{A_1\cosh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1) - A_2\sinh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1)}{A_1\sinh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1) + A_2\cosh\sqrt{-\frac{A}{2}}(x - \eta t + \xi_1)} \times \exp(i(kx + \omega t + \xi_0)) \quad (38)$$

It follows from (21)(22) and (38) that

$$n_9 = \frac{\beta\omega^2\Phi^2(\xi)}{k^2-\omega^2} = -\frac{\beta\omega^2}{k^2-\omega^2}\frac{2}{B}\left(\frac{A_1\cosh\sqrt{-\frac{A}{2}}(x-\eta t+\xi_1)-A_2\sinh\sqrt{-\frac{A}{2}}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{-\frac{A}{2}}(x-\eta t+\xi_1)+A_2\cosh\sqrt{-\frac{A}{2}}(x-\eta t+\xi_1)}\right)^2. \tag{39}$$

Case 2.3. From (15), (25) in **Case(vi)**, we get

$$\Phi = a_1\phi + b_1\psi = a_1\frac{G'}{G} + b_1\frac{1}{G}$$

and

$$u_{10} = \pm\sqrt{-\frac{1}{2B}}\frac{A_1\cosh\sqrt{-2A}(x-\eta t+\xi_1)-A_2\sinh\sqrt{-2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{-2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{-2A}(x-\eta t+\xi_1)}\mathrm{e}^{i(kx+\omega t+\xi_0)}$$
$$\pm\sqrt{-\frac{4A(A_1^2+A_2^2)-\mu^2}{4AB}}\frac{\exp(i(kx+\omega t+\xi_0))}{A_1\sinh\sqrt{-2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{-2A}(x-\eta t+\xi_1)} = \pm\sqrt{-\frac{1}{2B}}$$
$$\times\frac{\sqrt{\frac{\mu^2-4A(A_1^2+A_2^2)}{2A}}+A_1\cosh\sqrt{-2A}(x-\eta t+\xi_1)-A_2\sinh\sqrt{-2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{-2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{-2A}(x-\eta t+\xi_1)}\mathrm{e}^{i(kx+\omega t+\xi_0)} \tag{40}$$

It follows from (21), (22) and (40) that

$$n_{10}=-\frac{\beta\omega^2}{k^2-\omega^2}\frac{1}{2B}\left(\frac{\sqrt{\frac{\mu^2-4A(A_1^2+A_2^2)}{2A}}+A_1\cosh\sqrt{-2A}(x-\eta t+\xi_1)-A_2\sinh\sqrt{-2A}(x-\eta t+\xi_1)}{A_1\sinh\sqrt{-2A}(x-\eta t+\xi_1)+A_2\cosh\sqrt{-2A}(x-\eta t+\xi_1)}\right)^2. \tag{41}$$

Case 3. When $\lambda = 0$, by analogous computations, we obtain that if $A = 0$ and $B < 0$, then

$$a_1 = 0,\ b_1 = \pm\sqrt{\frac{-2}{B}},\ \mu = 0.$$

From the above case, we obtain the rational function solutions of (2) and (3). From (10), (15) and (25), we get

$$\Phi = b_1\psi = b_1\frac{1}{G}$$

and

$$u_{11} = \pm\sqrt{\frac{-2}{B}}\frac{A_1}{A_1(x-\eta t+\xi_1)+A_2}\exp(i(kx+\omega t+\xi_0)) \tag{42}$$

It follows from (21)(22) and (42) that

$$n_{11} = \frac{\beta\omega^2\Phi^2(\xi)}{k^2-\omega^2} = -\frac{\beta\omega^2}{k^2-\omega^2}\frac{2}{B}\left(\frac{A_1}{A_1(x-\eta t+\xi_1)+A_2}\right)^2. \tag{43}$$

Remark 3.3. Taking $A = -1,\ A = -2,\ A = -\frac{1}{2},\ \mu = -r$ and $\lambda = -1$, the solutions (26)-(31) are in full agreement with the corresponding solutions (74)-(75), (76)-(77) and (70)-(71) in Ref.[26]. But, the rational function solution (42)-(43) was not given in Ref.[26].

4. CONCLUSION AND DISCUSSION

In this paper, we propose an new approach for finding multiple exact solutions involving arbitrary parameters for the Klein-Gordon-Zakharov equations. By using this method and symbolic computation, we have found new types of exact solutions for the KGZs (2) and (3). When $\mu = 0$ in (5) and $b_i = 0$ in expansion (14), our proposed method is the $(\frac{G'}{G})$-expansion method. It is easy to check that the solutions (28),(29), (38),(39),(42) and (43) are in full agreement with the results obtained by using the $(\frac{G'}{G})$-expansion method.

We can also compare the our proposed method with other methods such as the extended hyperbolic function method. In the later method, the projective Riccati equations are chosen as its subsidiary ODE to construct the solutions of NLPDEs. Although in our contribution, we have seen that two variable $\phi = \frac{G'}{G}$ and $psi = \frac{1}{G}$ given in (2.2) also satisfy the projective Riccati equations $\phi' = -\phi^2 + \mu\psi - \lambda$, $\psi' = -\phi\psi$ given in (7), it is worthy of note that we do not use the special solutions of Eqs.(7) at all. Instead, we use directly the general solutions of the seconder order LODE (5), which is well known to researchers, to construct the solutions of NLPDEs. Thus, our proposed method has its own advantages: direct, concise and elementary. More importantly, we believe that this method can be used for many other NLPDEs in mathematical physics.

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