GENERALIZED PARAFERMIONIC OSCILLATORS IN PHYSICAL SYSTEMS

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Manifestations of generalized parafermionic oscillators in quantum superintegrable systems, as well as in selected cases in the structure of molecules, atomic nuclei, and bright solitons in Bose–Einstein condensates are discussed.

Key words: Generalized parafermionic oscillators, quantum superintegrable systems, nuclear pairing, bright solitons.

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1. INTRODUCTION

Elementary particles can be either bosons, for which each quantum state can be occupied by any number of particles up to infinity, or fermions, for which each quantum state can be occupied by only one particle because of the Pauli principle. However, composite bodies can present intermediate statistics [1], behaving as parafermions of order $p$, for which each quantum state can be occupied by up to $p$ particles. Clearly fermions are parafermions of order one, while bosons are parafermions of infinite order.

Parafermions can be conveniently described in the framework of generalized deformed oscillators [2–4], using a formalism similar to that of the usual harmonic oscillator in terms of creation and annihilation operators. In Section 2 we briefly review this formalism, while in the remainder of the paper we give several manifestations of generalized parafermionic oscillators in quantum superintegrable systems (Sections 3–7), molecules (Section 9), atomic nuclei (Section 10), and bright solitons in Bose–Einstein condensates (Section 11).
2. GENERALIZED DEFORMED PARAERMIONIC OSCILLATORS

A deformed oscillator [2–4] can be defined by the algebra generated by the operators \( \{1, a, a^+, N\} \) and the structure function \( \Phi(x) \), satisfying the relations

\[
[a, N] = a, \quad [a^+, N] = -a^+,
\]
and

\[
a^+ a = \Phi(N) = [N], \quad a a^+ = \Phi(N + 1) = [N + 1],
\]
where \( \Phi(x) \) is a positive analytic function with \( \Phi(0) = 0 \) and \( N \) is the number operator. From Eq. (2) we conclude that

\[
N = \Phi^{-1}(a^+ a),
\]
and that the following commutation and anticommutation relations are obviously satisfied

\[
[a, a^+] = [N + 1] - [N], \quad \{a, a^+\} = [N + 1] + [N].
\]

The structure function \( \Phi(x) \) is characteristic to the deformation scheme. For \( \Phi(x) = x \) the usual harmonic oscillator is obtained. Several examples can be found in Table 1 of Ref. [4].

It has been proved [5] that any generalized deformed parafermionic algebra of order \( p \) can be written as a generalized oscillator with structure function

\[
F(x) = x(p + 1 - x)(\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots),
\]
where \( \lambda, \mu, \nu, \rho, \sigma, \ldots \) are real constants satisfying the conditions

\[
\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots > 0, \quad x \in \{1, 2, \ldots, p\}.
\]

Using these generalized deformed parafermions as building blocks, one can consider the Bose-like Hamiltonian

\[
H = \frac{1}{2} \{a, a^\dagger\} = \frac{1}{2}(aa^\dagger + a^\dagger a),
\]
possessing the energy eigenvalues

\[
E(n) = \frac{1}{2}(F(n) + F(n + 1)), \quad n = 0, 1, 2, \ldots
\]
Using Eq. (5), and keeping terms up to \( n^4 \), the energy eigenvalues are written as

\[
E(n) = (\lambda + \mu + \nu)p + ((2\lambda + 2\mu + 3\nu)p - \mu - \nu)n

+ ((2\mu + 3\nu)p - 2\lambda - \mu - 3\nu)n^2 + (2\nu p - 2\mu - 2\nu)n^3 - 2\nu n^4.
\]
3. ISOTROPIC HARMONIC OSCILLATOR IN A 2D CURVED SPACE

Higgs [6] has studied the symmetries of a harmonic oscillator in a non-flat space, a space with constant curvature in particular. A typical example of such a space is the surface of the sphere in a three dimensional space.

The curved space is geometrically described by the metric
\[ ds^2 = \frac{dx^2 + dy^2 + \lambda(xdy - ydx)^2}{(1 + \lambda(x^2 + y^2))^2}, \] (10)
the flat space corresponding to \( \lambda = 0 \). The harmonic oscillator in this space is defined in Ref. [6] by the Hamiltonian:
\[ H = \frac{1}{2} \left( \pi_x^2 + \pi_y^2 + \lambda L^2 \right) + \frac{\omega^2}{2} \left( x^2 + y^2 \right), \] (11)
where \( L = xp_y - yp_x \) and
\[ \pi_x = p_x + \frac{\lambda}{2} \left( x (xp_x + yp_y) + (xp_x + yp_y) x \right), \]
\[ \pi_y = p_y + \frac{\lambda}{2} \left( y (xp_x + yp_y) + (xp_x + yp_y) y \right). \] (12)

The algebra of the isotropic harmonic oscillator in a 2D curved space with constant curvature \( \lambda \) for finite representations can be put in the form [7]
\[ F(N) = 4N(p+1-N) \left( \lambda(p+1-N) + \sqrt{\omega^2 + \lambda^2/4} \right) \left( \lambda N + \sqrt{\omega^2 + \lambda^2/4} \right), \] (13)
the relevant energy eigenvalues being
\[ E_p = \sqrt{\omega^2 + \frac{\lambda^2}{4} (p+1)} + \frac{\lambda}{2} (p+1)^2, \] (14)
where \( \omega \) is the angular frequency of the oscillator. It is clear that the condition of Eq. (6) is satisfied without any further restrictions.

4. THE KEPLER PROBLEM IN A 2D CURVED SPACE

The case of the Kepler problem in a space with constant curvature has been studied by Higgs [6]. The Hamiltonian is given by
\[ H = \frac{1}{2} \left( \pi_x^2 + \pi_y^2 + \lambda L^2 \right) - \frac{\mu}{r}, \quad r = \sqrt{x^2 + y^2}, \] (15)
where the operators \( L, \pi_x, \pi_y \) are the same as in the previous section.

The algebra of the Kepler problem in a 2D curved space with constant curvature \( \lambda \) for finite representations can be put in the form [7]
\[ F(N) = N(p+1-N) \left( \frac{4\mu^2}{(p+1)^2} + \lambda \frac{(p+1-2N)^2}{4} \right), \] (16)
the corresponding energy eigenvalues being

\[ E_p = -\frac{2\mu^2}{(p+1)^2} + \frac{\lambda(p+2)}{8}, \]  

(17)

where \( \mu \) is the coefficient of the \(-1/r\) term in the Hamiltonian. It is clear that the restrictions of Eq. (6) are satisfied automatically.

5. THE FOKAS–LAGERSTROM POTENTIAL

The Fokas–Lagerstrom potential [8] is described by the Hamiltonian

\[ H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{x^2}{2} + \frac{y^2}{18}. \]  

(67)

It is therefore an anisotropic oscillator with ratio of frequencies 3:1. For finite representations it can be seen [7] that the relevant algebra can be put in the form

\[ F(N) = 16N(p + 1 - N)\left(p + \frac{2}{3} - N\right)\left(p + \frac{4}{3} - N\right) \]  

for energy eigenvalues \( E_p = p + 1 \), or in the form

\[ F(N) = 16N(p + 1 - N)\left(p + \frac{2}{3} - N\right)\left(p + \frac{4}{3} - N\right) \]  

(18)

for eigenvalues \( E_p = p + 2/3 \), or in the form

\[ F(N) = 16N(p + 1 - N)\left(p + \frac{5}{3} - N\right)\left(p + \frac{4}{3} - N\right) \]  

(19)

for energies \( E_p = p + 4/3 \). In all cases it is clear that the restrictions of Eq. (6) are satisfied.

6. THE SMORODINSKY–WINTERNITZ POTENTIAL

The Smorodinsky–Winternitz potential [9] is described by the Hamiltonian

\[ H = \frac{1}{2}(p_x^2 + p_y^2) + k(x^2 + y^2) + \frac{c}{x^2}. \]  

(21)

i.e. it is a generalization of the isotropic harmonic oscillator in two dimensions. For finite representations it can be seen [7] that the relevant algebra takes the form

\[ F(N) = 1024k^2N(p + 1 - N)\left(N + \frac{1}{2}\right)\left(p + 1 + \frac{\sqrt{1+8c}}{2} - N\right) \]  

(22)

for \( c \geq -1/8 \) and energy eigenvalues

\[ E_p = \sqrt{8k}\left(p + \frac{5}{4} + \frac{\sqrt{1+8c}}{4}\right), \quad p = 1, 2, \ldots \]  

(23)
In the special case of \(-1/8 \leq c \leq 3/8\) and energy eigenvalues
\[ E_p = \sqrt{8k} \left( p + \frac{5}{4} - \frac{\sqrt{1 + 8c}}{4} \right), \quad p = 1, 2, \ldots \] \hspace{1cm} (24)
the relevant algebra is
\[ F(N) = 1024k^2 N(p + 1 - N) \left( N + \frac{1}{2} \right) \left( p + 1 - \frac{\sqrt{1 + 8c}}{2} - N \right). \] \hspace{1cm} (25)
In both cases the restrictions of Eq. (6) are satisfied.

7. THE HOLT POTENTIAL

The Holt potential [10]
\[ H = \frac{1}{2} (p_x^2 + p_y^2) + (x^2 + 4y^2) + \frac{\delta}{x^2} \] \hspace{1cm} (26)
is a generalization of the harmonic oscillator potential with a ratio of frequencies 2:1. The relevant algebra can be put [7] in the form of an oscillator with
\[ F(N) = 2^{3/2} N(p + 1 - N) \left( p + 1 + \frac{\sqrt{1 + 8\delta}}{2} - N \right), \] \hspace{1cm} (27)
where \((1 + 8\delta) \geq 0\), the relevant energies being given by
\[ E_p = \sqrt{8} \left( p + 1 + \frac{\sqrt{1 + 8\delta}}{4} \right). \] \hspace{1cm} (28)
In this case it is clear that the condition of Eq. (6) is always satisfied without any further restrictions.

In the special case \(-\frac{1}{8} \leq \delta \leq \frac{3}{8}\) one obtains [7]
\[ F(N) = 2^{3/2} N(p + 1 - N) \left( p + 1 - \frac{\sqrt{1 + 8\delta}}{2} - N \right), \] \hspace{1cm} (29)
the relevant energies being
\[ E_p = \sqrt{8} \left( p + 1 - \frac{\sqrt{1 + 8\delta}}{4} \right). \] \hspace{1cm} (30)
The condition of Eq. (6) is again satisfied without any further restrictions within the given range of \(\delta\) values.

8. TWO IDENTICAL PARTICLES IN TWO DIMENSIONS

Let us consider the system of two identical particles in two dimensions. For identical particles observables of the system have to be invariant under exchange of
particle indices. A set of appropriate observables in this case is \([11, 12]\)

\[
u = (x_1)^2 + (x_2)^2, \quad w = 2x_1x_2, \quad U = (p_1)^2 + (p_2)^2, \quad V = (p_1)^2 - (p_2)^2, \quad W = 2p_1p_2,
\]

(31)

where the indices 1 and 2 indicate the two particles. These observables are known to close an \(\text{sp}(4, \mathbb{R})\) algebra. A representation of this algebra can be constructed \([11]\) using one arbitrary constant \(\eta\) and three matrices \(Q, R, S\) satisfying the commutation relations

\[
[S, Q] = -2iR, \quad [S, R] = 2iQ, \quad [Q, R] = -8iS(\eta - 2S^2). \tag{34}
\]

The explicit expressions of the generators of \(\text{sp}(4, \mathbb{R})\) in terms of \(\eta, S, Q, R\) are given in \([11]\) and need not be repeated here. Defining the operators

\[
X = Q - iR, \quad Y = Q + iR, \quad S_0 = \frac{S}{2}, \tag{35}
\]

one can see that the commutators of Eq. (34) take the form

\[
[S_0, X] = X, \quad [S_0, Y] = -Y, \quad [X, Y] = 32S_0(\eta - 8(S_0)^2), \tag{36}
\]

which is a deformed version of \(\text{su}(2)\).

Using the same procedure as above, the algebra of Eq. (36) can be put in correspondence with a parafermionic oscillator characterized by

\[
F(N) = N(p + 1 - N)64(p + (1 - p)N - N^2), \tag{37}
\]

if the condition

\[
\eta = 4p(p + 1) \tag{38}
\]

holds. However, the condition of Eq. (6) is violated in this case.

### 9. THE MORSE POTENTIAL

The energy eigenvalues of the Morse potential \([13]\)

\[
V(x) = e^{-2ax} - 2e^{-ax}, \tag{39}
\]

are known to have the form \([14]\)

\[
E(n) = \left(n + \frac{1}{2}\right) - \alpha \left(n + \frac{1}{2}\right)^2, \tag{40}
\]

where \(a, \alpha\) are constants.

Trying to describe the Morse oscillator in terms of a Bose-like oscillator built with generalized deformed parafermions, one should equate the spectrum of Eq. (40)
with the spectrum of Eq. (9). Solving the relevant simple system of linear equations one obtains the conditions

\[ \lambda = \alpha, \quad \mu = \nu = 0, \quad p = \frac{1}{\alpha} - 1. \] (41)

We see that the coefficient \( \lambda \) and the order of the parafermions, \( p \), are determined in terms of the parameter \( \alpha \) appearing in the spectrum. Since \( p \) has to be a positive integer, only values of \( \alpha \) leading to such values of \( p \) can be considered.

A qualitative discussion of this result is in order. For Morse oscillators deviating little from the harmonic oscillator behavior, the parameter \( \alpha \) obtains small values, thus leading through Eq. (41) to large values of the order \( p \), corresponding to near-bosonic behavior. On the contrary, large deviations from the harmonic behavior correspond to larger values of \( \alpha \), leading to smaller values of \( p \), i.e., to behaviors closer to the fermionic one.

10. PAIRING IN A SINGLE-\( j \) NUCLEAR SHELL

In the usual formulation of the theory of pairing in a single-\( j \) shell [15], fermion pairs of angular momentum \( J = 0 \) are created by the pair creation operators

\[ S^+ = \frac{1}{\sqrt{\Omega}} \sum_{m>0} (-1)^{j+m} a^+_j a^+_{j-m}, \] (42)

where \( a^+_j \) are fermion creation operators and \( 2\Omega = 2j+1 \) is the degeneracy of the shell. In addition, pairs of nonzero angular momentum are created by the \( \Omega - 1 \) operators

\[ B_j^+ = \sum_{m>0} (-1)^{j+m} (j m j - m |J0) a^+_j a^+_{j-m}, \] (43)

where \( (j m j - m |J0) \) are the usual Clebsch Gordan coefficients. The fermion number operator is defined as

\[ N_F = \sum_m a^+_j a_{jm} = \sum_{m>0} (a^+_j a_{jm} + a^+_{j-m} a_{j-m}). \] (44)

The \( J = 0 \) pair creation and annihilation operators satisfy the commutation relation

\[ [S, S^+] = 1 - \frac{N_F}{\Omega}, \] (45)

while the pairing Hamiltonian is

\[ H = -G\Omega S^+ S. \] (46)
The seniority $V_F$ is defined as the number of fermions not coupled to $J = 0$. If only pairs of $J = 0$ are present (i.e. $V_F = 0$), the eigenvalues of the Hamiltonian are
\[ E(N_F, V_F = 0) = -G\Omega \left( \frac{N_F}{2} + \frac{N_F^2}{4\Omega} - \frac{N_F^2}{4\Omega} \right) \] (47)
For non-zero seniority the eigenvalues of the Hamiltonian are
\[ E(N_F, V_F) = -\frac{G}{4} (N_F - V_F)(2\Omega - N_F - V_F + 2). \] (48)
We denote the operators $N_F$, $V_F$ and their eigenvalues by the same symbol for simplicity.

For the case of nonzero seniority, one observes that Eq. (48) can be written as
\[ E(N_F, V_F) = G\Omega \left( \frac{V_F}{2} + \frac{V_F^2}{2\Omega} - \frac{V_F^2}{4\Omega} \right) - G\Omega \left( \frac{N_F}{2} + \frac{N_F}{2\Omega} - \frac{N_F^2}{4\Omega} \right), \] (49)
\textit{i.e.} it can be separated into two parts, formally identical to each other.

Let us first consider the case in which only $J = 0$ pairs are present. Equating the spectrum of Eq. (47) without the minus sign to Eq. (9), one finds
\[ \lambda = \frac{1}{8}, \quad p = 2\Omega + 2 = 2j + 3. \] (50)
Therefore, fermion pairs with $J = 0$ in a single-$j$ shell can be described as parafermions of order $p = 2j + 3$ through a Bose-like Hamiltonian. Since $j$ is half-integer, $p$ is an integer number.

Considering now the pairs with non-zero angular momentum, we see that the first term in Eq. (49), when equated to Eq. (9), gives again the results of Eq. (50). Thus Eq. (49) can be described as the difference of two parafermionic oscillators of order $p$.

11. BRIGHT SOLITONS IN BOSE–EINSTEIN CONDENSATES

Stable bright solitons have been recently constructed [16, 17] in a stable Bose–Einstein condensate of $^7$Li atoms cooled down at almost zero Kelvin temperature and trapped in a cylindrical, magneto-optical trap. The interaction among the lithium atoms has been tuned from repulsive to attractive through a strong magnetic field. The solitons were created when the interaction among the lithium atoms was attractive. The motion of the solitons, triggered by offsetting the optical potential, revealed repulsive interactions among neighboring solitons, although the interactions among the lithium atoms were attractive. In other words, solitons consisting of bosons with attractive interactions among themselves, exhibit a fermion-like behavior, repelling each other.
In order to justify this effect, we consider a dilute ultracold atomic gas composed by $N$ interacting bosons of mass $m$ \cite{18}. Every atom of the condensate is under the influence of the potential of the external, magneto-optical trap, $V_{\text{ext}}$, which has a cylindrical symmetry \cite{16}

$$V_{\text{ext}} = \frac{1}{2} m \omega_r^2 r^2 + \frac{1}{2} m \omega_z^2 z^2, \quad \omega_r \gg \omega_z,$$

and the potential describing the interaction among the atoms, $V_{\text{int}}$, which is assumed to be a delta interaction

$$V_{\text{int}}(\mathbf{r} - \mathbf{r}') = g \delta(\mathbf{r} - \mathbf{r}').$$

The coupling constant $g$ possesses negative values, since the atoms are tuned to attract each other.

The condensate is described by a nonlinear Schrödinger equation (NLS) named the Gross–Pitaevskii (GP) equation \cite{18, 19}

$$i \hbar \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left( \frac{-\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g |\Phi(\mathbf{r}, t)|^2 \right) \Phi(\mathbf{r}, t),$$

where $\Phi(\mathbf{r}, t)$ is the macroscopic wave function of the condensate, while the term $g|\Phi(\mathbf{r}, t)|^2$ is due to the effective interaction potential among the atoms and it is responsible for the nonlinearity of the above Schrödinger equation.

Eq. (53) is a nonlinear differential equation with no exact solution. However, in the present situation, an approximate bright soliton solution can be obtained taking into account the condition $\omega_r \gg \omega_z$ and the fact that the trapping potential varies slowly along the $z$ axis, leading to an effectively 1D system \cite{19, 20}

$$U(z, t) = A \eta \text{sech}[\eta(z - vt)] e^{i(kz - \omega t)},$$

where $\eta$, $v$, $\omega$, and $k$ represent the soliton inverse width, velocity, frequency and wavenumber respectively. The energy of the BEC is then found to be

$$E = \frac{\hbar^2 k^2}{2m} N + \frac{1}{12 A^2} \left( g_{1D} + \frac{\hbar^2}{2m A^2} \right) N^3,$$

where $g_{1D}$ is the effective interaction strength in one dimension. Comparison to Eq. (9) leads to

$$\Phi(N) = \mu N (p + 1 - N) \left( p - \frac{1}{2} + N \right),$$

with

$$\mu = -\frac{1}{12 A^2} \left( g_{1D} + \frac{\hbar^2}{2m A^2} \right), \quad p = -\frac{1}{4} + \sqrt{\frac{9}{16} + \frac{\hbar^2 k^2}{2m \mu}},$$

which are acceptable values, since $g_{1D} < 0$ for the attractive interactions considered.
here. In the limit of large $p$ a simplified form occurs

$$\Phi(N) = \mu N(p-N)(p+N) = \mu N(p^2 - N^2), \quad p = \frac{\hbar k}{\sqrt{2m\mu}}. \quad (58)$$

We conclude that interacting bosons within a Bose–Einstein condensate confined in one dimension can be described as parafermions, the order of the parafermions being related to the strength of the attractive interaction among the atoms. (No such representation is possible in the case of repulsive interactions among the atoms.) As a result, bright solitons within a Bose–Einstein condensate in one dimension exhibit repulsion among themselves, as a consequence of their parafermionic nature.

12. CONCLUSION

Generalized parafermionic oscillators appear to be a tool of wide applicability in the description of composite physical systems, including quantum superintegrable systems, molecules, atomic nuclei, and bright solitons in Bose–Einstein condensates.

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