INTERACTING FERMIONS IN NUCLEAR PHYSICS; FROM BOUND STATES TO FESHBACH RESONANCES

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A study of Feshbach resonances and the role of bound states in nuclear physics is presented. This work is an extension of parallel studies in cold atoms. The tuning in atomic physics is via a magnetic field while the nuclear case can be tuned by varying the proton fraction. The unitary limit of infinite scattering length is studied. Applications to the interaction energy, equation of state, compressibility, entropy, viscosity to entropy density are given.

Key word: Feshbach resonance, interacting fermions.

INTRODUCTION

This manuscript is devoted to the understand of the role of bound states and Feshbach resonances in strongly interacting nuclear systems started in [1, 2]. Such a study parallels that in atomic systems were recent studies [3-6] focused on the magnetically tuned transitions from a Bose-Einstein condensed state of bound fermionic pairs to a BCS superfluid state of unbound but spin correlated fermionic pairs. One of the important aspects of this investigate is the study of properties associated with the unitary limit were the scattering length \( a_i \) is very large compared to the range of the potential and the distance between nucleons. The unitary limit is associated with universal thermodynamics behavior. Another issue in this study is the role the effective range \( r_0 \) plays when the scattering length \( a_i \) goes to infinity. The \( S - \) wave phase shift \( \delta_0 \) in an effective range theory is \( k \cot \delta_0 = -1/a_i + r_0 k^2 / 2 \). The momentum is \( p = \hbar k \). The condition for a resonant state is \( a_i < 0 \) and for a bound state the \( a_i > 0 \). The effective range theory is a low energies approximation and higher order partial waves contribute at higher energies as mentioned below. Such studies are important for understanding the physics of
strongly correlated fermionic systems in nuclei, neutron stars, atomic physics, and the quark-gluon plasma. In atomic systems experiments are done in fermionic Li6 (Z=3) & K40 (Z=19). The scattering length is tuned with a magnetic field. In nuclear systems interesting differences with atomic systems appear which can add additional information. These differences are:

A. Nuclei are two component fermionic systems with protons & neutrons. Each component has spin up and spin down parts. Thus we have a four fermionic system. Atomic systems are of one type with spin up and spin down possibilities. The high temperature universal thermodynamic limit in atomic systems was studied in ref. [7, 8].

B. Nuclear interactions have an underlying isospin symmetry: \( V_{nn} \approx V_{np} \approx V_{pp} \). This useful fact allows us to study the role of symmetries in strongly correlated fermionic systems.

C. The nuclear interaction has a strong isospin dependent term, a spin-orbit force, a non-central tensor force and velocity dependent terms. The isospin dependent terms leads to a symmetry energy. The symmetry energy is vital in neutron star physics.

D. Nucleons can form complex bound states beyond mass 2 or the deuteron. A nuclear system can make a phase transition - a liquid gas phase transition [9].

Not only is the unitary limit of interest in itself, but it plays a significant role in the viscosity of interacting matter and in near perfect fluid behavior. Such behavior is recently reported in atomic systems [10] and relativistic heavy ion collision [11, 12]. Interest arose out of string theory considerations which gave a lower limit for the viscosity to entropy density ratio [13].

2. PROPERTIES OF NUCLEAR SYSTEMS
AND THE VIRIAL EQUATION OF STATE

Table 1 gives the isospin structure of the nucleon-nucleon system.

<table>
<thead>
<tr>
<th>Isospin symmetric I=1 states: ( pp ) ( nn ) ( np )</th>
<th>Isospin antisymmetric I=0 states: ( np )</th>
</tr>
</thead>
<tbody>
<tr>
<td>orbital ( L=0 ) ( S - ) wave space symmetric +</td>
<td>orbital ( L=1 ) ( P - ) wave space antisymmetric -</td>
</tr>
<tr>
<td>total spin ( S=0 ) spin antisymmetric</td>
<td>total spin ( S=1 ) spin symmetric +</td>
</tr>
<tr>
<td>resonant like virtual states</td>
<td>deuteron bound state</td>
</tr>
<tr>
<td>orbital ( L=2 ) ( D - ) wave space symmetric +</td>
<td>orbital ( L=2 ) ( D - ) wave space symmetric +</td>
</tr>
<tr>
<td>total spin ( S=0 ) spin antisymmetric</td>
<td>total spin ( S=1 ) spin symmetric +</td>
</tr>
</tbody>
</table>
The interaction used in this study is a square well potential with outer radius $R$ and with an inner hard core of radius $c$. The
\[ \delta_0 = \arctan\left(\frac{(kR/\alpha R)}{\tan(\alpha(R-c) - k(R-c) - kc}\right) \]
with \( \alpha^2 = k^2 + \alpha_0^2 \) and \( \alpha_0 = \sqrt{2\mu V_0/h^2} \). The $V_0$ is the depth of the potential. Limiting situations of this potential are: A) Pure hard core where \( R \to c, \alpha \to 0, \alpha \to k \) and \( \delta_0 = -kc \). This limit can be compare with a Lee–Yang hard sphere Bose gas and Fermi gases. B) Pure square well where \( c \to 0 \) which is used in atomic physics studies [7, 8]. The square well parameters $V_0, R, c$ can be fitted to effective range parameters [1, 2].

A virial expansion of the equation of state (EOS) is used. Virial expansions have also been used by Horowitz and Schwenk to study low density neutron matter [14] and for cluster formation for low density nuclear matter [15]. An early use of the virial expansion for such studies is in [16]. The degree of cluster formation depends on the density, temperature and proton fraction. Low density, high temperatures, low proton fraction favor very light clusters. The large binding energy of the alpha particle enhances the presence of alpha particles through a Boltzmann factor in binding energy. Here, only the second order virial coefficient will be considered and the EOS is then given by \( PV = kT(A - b_2 A^2 / V) \). The $P$ is the pressure, $V$ is the volume and $A$ is the nucleon number. The second order virial coefficient can be written as \( \hat{b}_2 = \hat{b}_{2,\text{exc}} + \hat{b}_{2,\text{int}} \) with \( \hat{b}_{2,\text{exc}} = -\lambda^4 \left(y^2 + (1 - y)^2\right) / 2^7/2 \) coming from exchange symmetry terms. The $y$ is the proton fraction. The $\hat{b}_{2,\text{int}}$ is the interaction part of $\hat{b}_2$ with

\[
\hat{b}_{2,\text{int}} = \sum_s \sum_{(i,j) \neq p,n} y_j y_i \frac{2S+1}{2^2} \frac{2^{3/2} \lambda^4}{k^2 T} \left\{ \exp\left(\frac{E_{g}(ij,S)}{k^2 T}\right) \right\} + \frac{1}{\pi} \sum_l (2l + 1) \int \frac{d\delta_{ij}(ij,S)}{dk} \exp(-h^2 k^2 / mk^2 T) dk \right\}
\]

(1)

The $\hat{b}_{2,\text{int}}$ has bound and continuum correlations. The $y_p = y$ and $y_n = (1 - y)$. The thermal wave length is $\lambda_T = h / \sqrt{2\pi mk^2 T}$. The $E_g(j,i,S)$ is the binding energy of a bound state. The $\delta_{ij}(ij,S)$ is the phase shift for the $l'$ th partial wave. The $\hat{b}_{2,\text{int}}$ can be used to study various thermodynamic quantities such as the following.

1. The interaction energy $E$. For an ideal gas the internal energy is independent of the volume. For a non-ideal gas the volume dependence of $E$ can be obtained from the relation

\[ E = \sum_i \delta_{ij}(ij,S) \]
\[ \frac{\partial E}{\partial V} = T \frac{\partial P}{\partial T} - P \] and is \[ E(V) \equiv E(\hat{b}_z) = (T \hat{\partial}_{\hat{b}_z} / \partial T) \hat{b}_z T A^2 / V. \] (2)

2. The isothermal compressibility. The isothermal compressibility is determined by the equation

\[ \kappa_r = -(1/V)(\partial V / \partial P)_T = \kappa_{r,\text{ideal}} / (1 - 2\hat{b}_z A / V) \] (3)

with \( \kappa_{r,\text{ideal}} = 1 / ((A / V) b T) \) the ideal gas compressibility.

3. The interaction entropy. The interaction entropy is

\[ S - S_{\text{id}} = S(\hat{b}_z) = k_b A^2 / V d(\hat{T}\hat{b}_z) / dT \] (4)

where the ideal entropy \( S_{\text{id}} \) is \( S_{\text{id}} = A k_b \ln(e^{5/2} V g_s / A \lambda^3_r) \). The \( g_s = 2S + 1 \) is the spin contribution.

4. The viscosity and the ratio of viscosity to entropy density will be discussed below.

Properties of \( \hat{b}_{2,\text{int}} \) involve an evaluation of a continuum correlation integral involving \( d\delta / d k \) for the \( l' \) th partial wave. For \( S - \) waves an effective range approximation gives

\[ B_s(S - \text{wave}) = \frac{1}{\pi} \int_0^\infty \frac{d\delta_0}{dk} \exp(-bk^2)dk = \]

\[ -\frac{a_0^2 (2a_{sl} - 3r_0)}{4(a_{sl}^2 - a_{sl} r_0)^{1/2}} \exp\left(\frac{b}{a_{sl}^2 - a_{sl} r_0}\right) \text{Erfc}\left(\frac{b}{\sqrt{4(a_{sl}^2 - a_{sl} r_0)}\sqrt{\pi}b}\right) \] (5)

where \( b = h^2 / mk^2 T = \lambda_r^2 / 2\pi \).

The limit \( r_0 \to 0, a_{sl} \to -\infty \) has \( B_s \to 1/2 \) for all \( T \) and this is the limit considered in [7, 8] in their atomic physics studies. For \( B_s = 1/2 \), the \( \hat{b}_{2,\text{int}} = (1/4) 2^{3/2} \lambda_r^3 / 2 = \lambda_r^3 / 2^{3/2} \) from Eq. (1) for a spin 0 system as in pure neutron matter. The effective range corrections lead to departures from the value 1/2. For example, in the universal thermodynamic limit with \( E_\beta \to 0 \) and all \( |a_{sl}| \to \infty \), the \( \hat{b}_z / \lambda_r^3 \) to lowest order in \( r_0 / \sqrt{b} \) is

\[ \hat{b}_z / \lambda_r^3 \approx \left( \frac{3}{2^{3/2}} - \frac{2^{3/2}}{4} \frac{1}{6 \sqrt{\pi} \sqrt{b}} \right) - y(1-y) \left( \frac{3}{2^{3/2}} - \frac{2^{3/2}}{4} \frac{1}{6 \sqrt{\pi} \sqrt{b}} \right) \] (6)

The \( \hat{b}_z \) includes anti-symmetry exchange correlations and the above formula is for an isospin symmetric case with all effective ranges taken to be the same including
$a_{q_p,0}$ & $a_{q_p,1}$ for simplicity in the spin 0 and spin 1 channels. For pure neutron matter the proton fraction $y = 0$ and $\hat{b}_2 / \lambda_T^3 = (-1 + 4) / 2^{7/2}$ with the $-1 / 2^{7/2}$ coming from $\hat{b}_{2,unb}$. A rescaled interaction energy density is $\hat{\epsilon}_{\text{int}} = \epsilon_{\text{int}} / ((3k_B T / 2)(A/V)^2 \lambda_T^2 2^{3/2} / 4)$ and arises from the $\hat{b}_{2,unb}$ part of $\hat{b}_2$. Results for $\hat{\epsilon}_{\text{int}}$ (MeV) versus $y$ are shown in Fig. 1.

Fig. 1 – Left figure. The $\hat{\epsilon}_{\text{int}}$ versus $y$ for various $T$. The temperatures are $k_B T = 2.59 (b = 16)$, 3.45 ($b = 12$), 5.17 ($b = 8$). Deeper curves have lower $k_B T$. Lowering $T$ increases the importance of the bound state because of the Boltzmann factor. Right figure. The $\hat{\epsilon}_{\text{int}}$ versus $y$ and the role of a deuteron bound state and $T$ dependences. The right figure has $k_B T = 3.45$ or $b = 12$. In the right figure, the upper curve does not have a deuteron bound state. Instead a resonant state with parameters $a_d = -23.7$, $r_0 = 2.73$ was used for it. The deepest curve has a deuteron bound and is the same as the middle curve in the left figure. The middle line in the right figure is the unitary limit with $E_n = 0$ which is also the same as the unitary limit with an unbound or virtual deuteron. The figure shows the importance of the deuteron bound state on the interaction energy.

Interaction terms in the unitary limit with $a_d \to -\infty$ for pure neutron matter give:

$$B_C = \frac{1}{2} - \frac{1}{4\sqrt{\pi}} \frac{r_0}{\sqrt{b}} + \frac{1}{32\sqrt{\pi}} (\frac{r_0}{\sqrt{b}})^3 = \frac{1}{2} - \frac{\sqrt{2}}{4} \frac{r_0}{\lambda_T} + \frac{3\sqrt{2\pi}}{16} (\frac{r_0}{\lambda_T})^3$$

(7)

The following expression is for the energy.

$$\frac{E(\hat{b}_{2,\text{int}})}{V} \frac{1}{(A^2 / V^2) k_B T (2^{3/2} / 2^2) \lambda_T^3} = \frac{3}{2} \left( \frac{1}{2} - \frac{\sqrt{2}}{6} \frac{r_0}{\lambda_T} \right)$$

(8)

while the entropy can be obtained from
The interaction energy of Eq. 8 has a \( r_o / \kappa_T \) term but no \( (r_o / \kappa_T)^2 \) and \( (r_o / \kappa_T)^3 \) terms.

The interaction entropy \( S(\hat{b}_{2,\text{int}}) \) of Eq. 9 has no \( r_o / \kappa_T \) and \( (r_o / \kappa_T)^2 \) but has a \( (r_o / \kappa_T)^3 \) term. Thus, the effective range corrections are much larger for the energy than the entropy.

Effective range corrections to the ideal gas compressibility

\[
\kappa = \frac{2}{\pi} \frac{\lambda}{T} \frac{\rho}{\rho_0^3} \frac{1}{1 - \frac{3}{2^{5/2}} \frac{A}{V} \frac{\rho}{\rho_0} \frac{r_o}{4\lambda_T} \frac{3\sqrt{2}}{16} \left( \frac{r_o}{\lambda_T} \right)^3} > \kappa_{\text{ideal}}
\]

Since \( \kappa_T > \kappa_{\text{ideal}} \), the \( \kappa_T \) is above the ideal gas limit and this implies that \( \kappa_T \) will have a peak. Neglecting \( r_o / \kappa_T \) corrections, the

\[
\kappa_T = \xi_\kappa \kappa_{\text{ideal}}
\]

with

\[
\xi_\kappa = 1/(1 - (3/2^{5/2})(A/V)\lambda_T^3)
\]

In this limit properties of \( \kappa_T \) no longer contains features related to the potential.

The above results are limited to \( S \)–wave interactions. Higher partial waves become important when \( k_B T \geq 15 \text{MeV} \). Also in neutron-proton systems larger clusters beyond the deuteron can dominate the virial expansion as noted early on in [16] and in [15]. The sum over many partial waves for hard sphere scattering at very high energies leads to interesting scaling results. First the partial wave expansion for the scattering cross section \( \sigma \) from a hard sphere is given by the result [17]

\[
\sigma = (4\pi/k^2)\sum_{l=0} j_l^2(kc) + \eta_l^2(kc)
\]

At high energies this cross section goes to the well known scaling limit result \( \sigma \to 2\pi c^2 \) which is twice the classical geometric value of \( \sigma = \pi c^2 \) for scattering from a hard sphere. The extra two comes from diffractive scattering of the incident wave. A scaling behavior for the quantity
\[ \Sigma_{i=1}^{\infty} (2l+1)d\delta_i / dk = -\left( c/(ke)^2 \right) \Sigma_{i=0}^{\infty} (2l+1)/(j_i^2 (kc) + \eta_i^2 (kc)) \], which appears in the continuum correlation function, also exists \[1, 2\]. Defining two sums, \[ S_2 \equiv \Sigma_{i=0}^{\infty} (2l+1)/(j_i^2 (x) + \eta_i^2 (x)) \] and \[ S_i \equiv \Sigma_{j=0}^{\infty} (2l+1)/j_i^2 (x)/(j_i^2 (x) + \eta_i^2 (x)) \] with \[ x = kc \], the following scaling behaviors are found. The \[ S_2 \rightarrow 2\lambda^4 / 3 \] and \[ S_1 \rightarrow 1\lambda^2 / 2 \]. The \( \sigma \) is proportional to \( S_1 \). Using the scaling behavior for \( S_2 \) for \( x = kc >> 1 \), the following equation is obtained.

\[ \frac{1}{\pi} \int dk \sum_{l=0}^{\infty} (2l+1) \frac{d\delta_l}{dk} \exp(-bk^2) = -\frac{\sqrt{2}}{3} \pi \frac{c^3}{\lambda^3} \] \( (14) \)

This result shows that the second virial coefficient when summed over many partial waves at high energies depends on the radius of the hard core as \( c^3 \) and thus the volume of a hard sphere. The EOS acquires a van der Waals excluded volume term.

4. Viscosity and viscosity to entropy density; comparisons with string theory.

The simplest expression for the coefficient of shear viscosity \[ [18] \] is

\[ \eta = \frac{1}{3} \rho m \bar{\nu} l \] \( (15) \)

where \( \rho \) is the number density, the \( m \) is the mass of a fluid particle, \( \bar{\nu} = \sqrt{8k_B T / \pi m} \) is the mean speed and \( l = 1 / \rho \sigma \) is the mean free path. The \( \sigma \) is the cross which for hard sphere scattering is \( \pi D^2 \) with \( D \) the diameter of a fluid particle. We use this as a lowest order approximation as a basis for comparison. Using the Sackur-Tetrode expression for the entropy density

\[ s_{id} = S_{id} / V = \rho k_B \ln(e^{1/2} g_s V / A\lambda_\perp^4) \] \( (16) \)

Next we minimize the viscosity to entropy density ratio \( \eta / s_{id} \) with respect to lowest order ideal gas value of the fugacity \( z = \lambda_\perp^2 \rho \) at fixed \( y = Dp^{1/3} \equiv D / l_s \). The \( l_s = (V / A)^{1/3} \) is the spacing between particles. The minimum occurs at

\[ z = g_s / \sqrt{e} = (2s + 1) / \sqrt{e} \] \( (17) \)

It should be noted that the spin degeneracy factor pushes the minimum to higher values of the ideal gas fugacity. The minimum value of \( \eta / s_{id} \) is

\[ (\eta / s_{id})_{min} = \frac{4}{9\pi} \frac{e^{1/6}}{g_s^{1/3}} \frac{1}{y^2} \frac{1}{k_B} = 16 \frac{e^{1/6}}{9} \frac{1}{g_s^{1/3}} \frac{1}{y^2} \frac{1}{4\pi} \frac{1}{k_B} \] \( (18) \)

where the term \((1 / 4\pi)(h / k_B)\) is the string theory result \[13\]. The \( \eta / s_{id} \) varies as \( \eta / s_{id} \sim 1 / y^2 \). The higher the density the larger the value of \( \eta \) (the diameter to
spacing ratio) and the smaller the viscosity to entropy density. The most extreme limit for \(y\) is when the particles just touch which has \(y = 1\)

\[
\left(\frac{\eta}{s_{id}}\right)_m = \frac{16}{9} \frac{e^{3/2}}{g_{s}^{1/3}} \left(\frac{1}{4\pi} \frac{h}{k_B}\right) = \frac{2.1}{g_{s}^{1/3}} \left(\frac{1}{4\pi} \frac{h}{k_B}\right)
\]  

(19)

At this point the simplest theory would fail since higher order corrections to the viscosity and entropy would be present.

The next level of approximation comes from molecular dynamics studies \([19]\) for the viscosity and equation of state for a hard sphere spinless system. The equation of state can be used to obtain the entropy density using a Maxwell relation. Defining the packing fraction \(\xi = (\pi/6)D^3\rho = (\pi/6)y^3\) the equation of state departs from an ideal gas law and reads \([19]\)

\[
P = \rho k_B T \frac{1 + \xi + \xi^2 - \xi^3}{(1 - \xi)} = \rho k_B T(1 + 4\xi + 10\xi^2 + 18\xi^3 + ...)
\]

(20)

The associated entropy is obtained from the Maxwell relation \((\partial P/\partial T)_r = (\partial S/\partial V)_r\). Adding to this the Sackur-Tetrode ideal gas expression gives \([20]\)

\[
s = \rho k_B \left(\ln \frac{e^{5/2}}{z} \right) - 4\xi (1 - (3/4)\xi) \frac{(1 - \xi)}{(1 - \xi)^2}
\]

(21)

The viscosity was determined as \([19]\)

\[
\eta = \frac{5}{16} \frac{(\pi mk_B T)^{1/2}}{\pi D^2} \left[1.016 + 0.6600\xi + 14.1570\xi^2 + 30.82050\xi^3 + ...\right]
\]

(22)

The prefactor on the right side of the bracket is the Chapman-Enskog result \([21]\) for scattering off a hard sphere potential of radius \(D\). As before, but here excluding spin, the minimum with respect to temperature occurs at the ideal gas fugacity \(z = 1/\sqrt{e}\). The resulting \(\eta/s\) is then

\[
\eta/s = \frac{5\sqrt{2} e^{1/6}}{48} \frac{1 + .666\xi + 14.157\xi^2 + 30.82\xi^3}{(6/\pi)\xi^{2/3}} \frac{h}{(k_B)}
\]

(23)

The denominator factor \(((6/\pi)\xi)^{2/3} = y^2\) so that \(\eta/s \sim 1/y^2\) as before. The difference in the numerical prefactors \(5\sqrt{2} e^{1/6}/48\) and \(4e^{1/6}/9\pi\) is just the ratio of the initial viscosity expressions, the Chapman-Enskog prefactor factor \(5\sqrt{\pi}/16\) and the prefactor \((1/3)\sqrt{8/\pi}\). The last expression can be varied with respect to the
packing factor $\xi$. A minimum value occurs at $\xi = 0.07$ and thus $y = 0.5$ so that the spacing between particles is twice their diameter. The $\eta/s$ is

$$\frac{\eta}{s} = 1.08 \frac{h}{k_B} = 13.57 \frac{1}{4\pi} \frac{h}{k_B}$$  \hspace{1cm} (24)$$

The last result shows that $\eta/s$ is 13.57 times the string theory result. Most of the increase from the simpler approach which has 2.1 times the string theory result comes from the $1/y^2$ behavior which for $y = 0.5$ gives $1/y^2 = 4$.

The next question is: how is the viscosity modified by quantum theory and in particular the unitary limit of Feshbach resonances were the scattering length becomes infinite? For a pure neutron gas, which has a large scattering length, the limit was considered in ref. [20]. The quantum aspects of the viscosity where studied in ref. [22-25] for various other systems. The viscosity with scattering length $a_d$ and effective range $r_0$ corrections was shown to be given by the expression

$$\eta = \left[ \frac{a_d(a_d - r_0)}{a_d^2} + \frac{\lambda_f^2}{6\pi a_d^2} \right] \left( \frac{15}{16} \sqrt{2\pi} \frac{1}{\lambda_f} \right) h$$  \hspace{1cm} (25)$$

The unitary limit is obtained from the limit $a_d \to \infty$ giving

$$\eta(a_d = \infty) \to \left( \frac{15}{16} \sqrt{2\pi} \frac{1}{\lambda_f} \right) h$$  \hspace{1cm} (26)$$

In this limit the $1/\lambda_f^2$ behavior results in a $T^{5/2}$ behavior for the viscosity. The classical limit is obtained by taking the limit $h \to 0$. In this limit the second term in the square bracket survives so that

$$\eta(h = 0) \to \left[ \frac{1}{6\pi a_d^2} \left( \frac{15}{16} \sqrt{2\pi} \frac{1}{\lambda_f} \right) h \right] = \frac{5}{16\pi a_d^2} \left( \frac{1}{2} \sqrt{\pi mk_BT} \right)$$  \hspace{1cm} (27)$$

The extra factor of $1/2$ in this expression comes from anti-symmetry and spin effects. The temperature dependence is now $T^{1/2}$. For hard sphere scattering $a_d = D$. If the unitary limit is used for the viscosity and the entropy of an ideal gas is used to obtain the ratio $\eta/s$ the result is

$$\frac{\eta}{s} = \frac{15}{16} \sqrt{2\pi} \frac{1}{z(5/2 - \ln(z/g_s))} \frac{h}{k_B}$$  \hspace{1cm} (28)$$
The minimum of $\eta/s$ with respect to variations of $z$ is easily found and is $z = g_s e^{1.5} > 1$. The high value of the ideal gas fugacity is in a regime where the simple expression for the entropy is incomplete. Including a first order correction to the entropy density gives

$$s = p_k \left\{ \ln \left[ e^{\pi/2} g_s / z \right] + (1 - 4) \frac{z}{2^{7/2} g_s^2} \right\}$$

(29)

In the factor $(1-4)$ appearing in the last term, the 1 arises from anti-symmetry and the 4 is the universal thermodynamic limit or unitary limit where effective range terms are neglected. In this case the solution for $z$ has become somewhat smaller with $z = g_s 1.75$ but still greater than 1 so the same remarks apply. The unitary limit for the minimum of the viscosity to entropy ratio is therefore in a regime beyond the scope of the present calculation. We therefore leave it as an unsolved problem.

**CONCLUSIONS AND SUMMARY**

Nuclei offer the possibility for studies of strongly correlated fermionic systems using results obtained from a long history of research on nucleon-nucleon interactions. The interactions are such that both Feshbach resonances in the isospin1 channel (pp, nn & np virtual or resonant like states) with large scattering length and a weakly bound state (the deuteron) in the isospin 0 channel exist. Thus, almost ideal conditions exist for a study of the unitary limit of infinite scattering length or zero bound state energies. Therefore, a hope of this study is to gain universal insights useful also for neutron stars, atomic systems, and a strongly correlated quark-gluon system.

Nuclear systems differ somewhat from atomic systems. The nuclear system has two types of particles-protons and neutrons- each with two spin states. An underlying symmetry associated with isospin conservation is present. Understanding the role of symmetries has been a vital endeavor in many areas of physics. Isospin symmetry is one of the best examples with very little symmetry breaking. Atomic systems are tuned across the Feshbach resonance by varying an external magnetic field. In nuclear systems a hybrid type of tuning can be done by varying the proton fraction. In pure neutron systems a virtual resonant like state exists. Adding a small proton fraction introduces a bound deuteron state. A study of the unitary limit showed that the ratio $r_0 / \lambda_T$ of the effective range to quantum thermal wavelength appears as a limiting scale in various thermodynamic quantities. Corrections to the unitary limit $a_{nl} \rightarrow -\infty$ and also the limit $r_0 \rightarrow 0$ were developed and shown in Fig. 1. The interaction energy, entropy, and
isothermal compressibility have different dependences on $r_0 / \lambda_T$, with the entropy the least sensitive involving $(r_0 / \lambda_T)^3$. The isothermal compressibility is above the ideal gas limit and therefore has a peak or cusp. In the universal limit of $a_{nj} \to \pm \infty$ and neglecting $r_0 / \lambda_T$ corrections the isothermal compressibility for neutron matter is $\kappa_T = \xi \kappa_{T,\text{ideal}}$ with $\xi$ a universal function.

Studies using a hard sphere Lee-Yang model have an interesting scaling result involving the ratio of the hard sphere volume to quantum volume $\lambda_T^3$. The result parallels the increase in geometric cross section for scattering from a hard sphere at high energy. This increase is due to diffractive scattering around the sphere. The resulting equation of state has an excluded volume van der Waals component.

This paper also focuses on the viscosity and viscosity to entropy density ratio. The importance of the latter ratio came from string theory which gave a minimum value for it which involves Planck’s and Boltzmann’s constant. The minimum value is used to define a perfect liquid. Perfect liquid properties are being explored in both atomic and nuclear systems. This paper gave a simple expression for the viscosity that contains both the unitary limit and the hard sphere scattering limit. It was pointed out that the attempt to find a minimum value in $\eta / s$ leads to a condition on the ideal gas fugacity $(z = \lambda_T^3 \rho > 1)$ which showed that higher order effects are needed. Also discussed was a molecular dynamics result which has a minimum at low packing fraction but has a value for $\eta / s$ that is more than an 10 times the string theory result.

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