We describe $\alpha$-decay transitions to excited states in even-even nuclei within the Coherent State Model (CSM). This formalism is able to simultaneously describe electromagnetic and $\alpha$-decays to excited states in spherical, transitional and well deformed nuclei. The $\alpha$-daughter interaction contains the monopole potential, estimated within the double folding procedure with M3Y interaction plus a repulsive core simulating Pauli principle and a quadrupole-quadrupole (QQ) interaction. The decaying states are identified with the lowest narrow outgoing resonances in this potential. The $\alpha$-branching ratios to $2^+$ states were reproduced by using the QQ strength depending linearly on the deformation parameter, as predicted by CSM. The predicted intensities to $4^+$ and $6^+$ states are in a reasonable agreement with available experimental data.

Key words: coherent state model, $\alpha$-transitions, coupled channels approach.

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1. INTRODUCTION

The relative values of $\alpha$-decay half-lives can be satisfactorily described within the Gamow penetration picture of a preformed $\alpha$-particle through the Coulomb barrier [1]. In order to describe absolute half lives it is also necessary to consider the $\alpha$-particle formation probability within the R-matrix theory [2–4]. For transitions between ground states the decay width is not sensitive to the nuclear structure details [5, 6]. The situation becomes different for transitions to excited states, where decay widths to excited states are very sensitive to the structure of the wave function in the daughter nucleus [7].

The first attempts to estimate Hindrance Factors (HF) in vibrational nuclei within the Quasiparticle Random-Phase Approximation (QRPA) were performed in Refs. [8, 9]. More recently, the fine structure of $2^+$ states was analyzed in Refs. [10–12] within the QRPA formalism.

The first computations of the $\alpha$-decay widths in rotational nuclei within the coupled channels approach were performed in Ref. [13]. There are many high precision data concerning $\alpha$-decay fine structure to excited states in even-even nuclei,
They were analyzed within the coupled channel formalism in Refs. [16, 17]. Later on, several papers were devoted to the coupled channel analysis of the $\alpha$-decay fine structure [18, 19], by using the double folding potential together with the Wildermuth rule to simulate the Pauli principle [20].

The aim of this paper is to give a description of $\alpha$-transitions to excited states in vibrational, transitional and well deformed nuclei, by using a common formalism provided by the Coherent State Model [21, 22].

The paper is organized according to the following plan. In Section II we describe the main theoretical ingredients and in Section III we perform an analysis of energies, electromagnetic transitions and $\alpha$-decay widths to excited states in even-even nuclei. In the last Section we draw conclusions.

2. COHERENT STATE MODEL DESCRIPTION OF ELECTROMAGNETIC AND $\alpha$-TRANSITIONS

A coherent superposition of boson operators describing surface vibrations of a deformed nucleus was discussed in Refs. [23], but a complete approach describing ground, $\beta$ and $\gamma$ bands was proposed in Refs. [21, 22] as the Coherent State Model (CSM) and it was extensively developed in Refs. [24, 25]. A recent review on this matter can be found in Ref. [26].

The wave function describing the intrinsic ground state is given by the following coherent superposition of quadrupole boson operators $b_{2\mu}$ with $\mu = 0$

$$|\psi_g\rangle = e^{d (b_{20}^\dagger - b_{20})} |0\rangle,$$

in terms of a deformation parameter, proportional to the static quadrupole deformation $d = \kappa \beta_2$ [24]. Physical states, defining the ground band, can be obtained by projecting out the angular momentum

$$\varphi_j^{(g)} = \mathcal{N}_j^{(g)} P_J^0 |\psi_g\rangle,$$

The key ingredient is the norm of the wave function, which can be estimated by using the following relations [25]

$$\mathcal{N}_j^{(g)} = (2J + 1) \left[ I_j^{(0)}(d) \right]^{-1/2} e^{d^2/2}, \quad I_j^{(0)}(d) = \int_0^1 P_j(x) e^{-x^2} P_2(x) \, dx,$$

where $P_J(x)$ is the Legendre polynomial. The expectation value of the number of bosons operator on the wave functions (2) is given by

$$\langle \varphi_j^{(g)} | \hat{N} | \varphi_j^{(g)} \rangle = \langle \varphi_j^{(g)} | \sum_{\mu} b_{2\mu}^\dagger b_{2\mu} | \varphi_j^{(g)} \rangle = d^2 \mathcal{I}_J(d),$$
in terms of the following universal function depending on deformation
\[ I_j(d) \equiv \frac{I_j^{(1)}(d)}{I_j^{(0)}(d)}, \quad I_j^{(1)}(d) = \frac{dI_j^{(0)}(x)}{dx}, \quad x = d^2. \] (5)

Notice that the ground band energy spectrum
\[ E_j(d) = \langle \phi_j^{(g)} | \hat{N} | \phi_j^{(g)} \rangle - \langle \phi_0^{(g)} | \hat{N} | \phi_0^{(g)} \rangle = d^2 [I_j(d) - I_0(d)], \] (6)
has a harmonic behaviour for small values of \( d \), while for \( d > 3 \), it has a rotational shape [25]. By using the quadrupole transition operator
\[ T_{2\mu} = q_0 Q_{2\mu}, \quad Q_{2\mu} = b_2^{(1)} b_2^{(2)} + b_2^{(2)} b_2^{(1)} + a_q \left( b_2^{(1)} \otimes b_2^{(2)} \right)_{2\mu} + (b_2 \otimes b_2)_{2\mu}, \] (7)
in terms of the effective charge \( q_0 \) and anharmonic strength \( a_q \), the B(E2) value for transitions between ground band states is given by [25]
\[ B(E2 : J' \rightarrow J) = \left| q(d) \langle J'0; 20 | J0 \rangle d \left( \frac{j_i N_i^{(g)}}{J_i^{(g)}} + \frac{j_f N_f^{(g)}}{J_f^{(g)}} \right) \right|^2, \] (8)
where we introduced the effective charge
\[ q(d) = q_0 \left( 1 - \frac{\sqrt{2}}{7} a_q d \right). \] (9)

Let us consider an \( \alpha \)-decay process \( P \rightarrow D(J) + \alpha \) where \( J \) denotes the spin of the excited state in the even-even axially deformed daughter nucleus. The wave function of the \( \alpha \)-daughter system has the total spin of the initial ground state (i.e. zero)
\[ \Psi(b_2, R) = \sum_j \frac{f_j(R)}{R} Z_j(b_2, \Omega) \] (10)
where
\[ Z_j(b_2, \Omega) \equiv \left[ \phi_j^{(g)} \otimes Y_j(\Omega) \right]_0. \] (11)
Here, \( R \equiv (R, \Omega) \) denotes the distance between the centers of two fragments. We describe the \( \alpha \)-daughter dynamics by using the stationary Schrödinger equation, i.e.
\[ H \Psi(b_2, R) = E \Psi(b_2, R). \] (12)
The Hamiltonian describing the \( \alpha \)-decay is written as follows
\[ H = -\frac{\hbar^2}{2\mu} \nabla_R^2 + H_D(b_2) + V(b_2, R), \] (13)
where \( \mu \) is the reduced mass of the dinuclear system and
\[ V(b_2, R) = V_0(R) + V_2(b_2, R), \] (14)
The monopole part of the interaction is given by the same ansatz as in Ref. [16], i.e.

\[ V_0(R) = \nabla_0(R), \quad R > R_m; \quad \Delta(R - R_{\text{min}})^2 - v_0, \quad R \leq R_m, \]  

(15)

where \( \nabla_0 \) is the nuclear plus Coulomb interaction, estimated by using the double folding procedure within the M3Y particle-particle interaction with Reid soft core parametrisation [27, 28]. The second line is the repulsive core simulating the Pauli effect (because the \( \alpha \)-particle can exist only on the surface) and fixing the energy of the first resonant state to the experimental \( Q \)-value [16]. The \( \lambda = 2 \) term is given by the quadrupole-quadrupole (QQ) interaction

\[ V_2(b_2, R) = -C_0(R - R_{\text{min}}) \frac{dV_0(R)}{dR} \frac{2}{\delta} [Q_2 \otimes Y_2(\Omega)]_0. \]  

(16)

By using the orthonormality of angular functions in (10) one obtains the coupled system of differential equations for radial components [29]

\[ \frac{d^2 f_J(R)}{d\rho_J^2} = \sum_{J'} A_{J,J'}(R) f_{J'}(R), \]  

(17)

where the coupling matrix is given by

\[ A_{J,J'}(R) = \left[ \frac{J(J+1)}{\rho_J^2} + \frac{V_0(R)}{E - E_J} - 1 \right] \delta_{J,J'} + \frac{1}{E - E_J} \langle Z_J|V_2(b_2, R)|Z_{J'} \rangle, \]  

(18)

in terms of the reduced radius

\[ \rho_J = \kappa_J R, \quad \kappa_J = \sqrt{\frac{2\mu(E - E_J)}{\hbar^2}}. \]  

(19)

The matrix element of the particle-core coupling is [21]

\[ \langle Z_J|V_2(b_2, R)|Z_{J'} \rangle = C(d)V_2(d), \]  

(20)

where

\[ V_2(d) = -(R - R_{\text{min}}) \frac{dV_0(R)}{dR} \frac{d}{dR} \frac{j}{\sqrt{4\pi} J'} (J_0|20, J'0)^2 \left( \frac{jN_{J'}^{(g)}}{jN_{J'}^{(g)}} + \frac{jN_{J'}^{(g)}}{jN_{J'}^{(g)}} \right), \]  

(21)

in terms of the effective \( \alpha \)-daughter coupling strength

\[ C(d) = C_0 \left( 1 - \sqrt{\frac{2}{7}} a_{\alpha} d \right). \]  

(22)

An \( \alpha \)-decaying state can be identified with a narrow resonant solution of the system (17), containing only outgoing components. Let us first define the internal and external fundamental solutions satisfying the following boundary conditions, respectively

\[ \mathcal{R}_{J,J}(R) \xrightarrow{R \to R_0} \delta_{J,J} \varepsilon_J, \]  

(23)
Here, $R_0$ is a radius inside the internal repulsive potential and $\varepsilon_J$ are arbitrary small numbers, $G_J(\kappa_J R), F_J(\kappa_J R)$ are the irregular and regular spherical Coulomb wave functions, respectively, depending on the momentum $\kappa_J$ in the channel $J$. Each component of the solution is a superposition of $N$ independent fundamental solutions. We impose the matching conditions at some radius $R_1$ inside the barrier

$$f_J(R_1) = \sum_I \mathcal{R}_{JJ}(R_1) M_I = \sum_I \mathcal{H}_{JJ}^{(+)}(R_1) N_I$$

(25)

$$\frac{df_J(R_1)}{dR} = \sum_I \frac{d\mathcal{R}_{JJ}(R_1)}{dR} M_I = \sum_I \frac{d\mathcal{H}_{JJ}^{(+)}(R_1)}{dR} N_I .$$

(26)

These conditions give the following secular equation

$$\begin{vmatrix}
\mathcal{R}(R_1) & \mathcal{H}_{JJ}^{(+)}(R_1) \\
\frac{\partial \mathcal{R}(R_1)}{\partial R} & \frac{\partial \mathcal{H}_{JJ}^{(+)}(R_1)}{\partial R}
\end{vmatrix} \approx \begin{vmatrix}
\mathcal{R}(R_1) & \mathcal{G}(R_1) \\
\frac{\partial \mathcal{R}(R_1)}{\partial R} & \frac{\partial \mathcal{G}(R_1)}{\partial R}
\end{vmatrix} = 0 .$$

(27)

The unknown coefficients $M_I, N_I$ are determined from the normalisation of the wave function in the internal region

$$\sum_J \int_{R_0}^{R_2} |f_J(R)|^2 dR = 1,$$

(28)

where $R_2$ is the external turning point. From the continuity equation one obtains the total decay width as a sum of partial widths

$$\Gamma = \sum_J \Gamma_J = \sum_J \hbar v_J \lim_{R \to \infty} |f_J(R)|^2 = \sum_J \hbar v_J |N_J|^2,$$

(29)

where $v_J$ is the center of mass velocity at infinity in the channel $J$.

3. ANALYSIS OF ENERGY SPECTRA AND $\alpha$-DECAY WIDTHS

We will analyse the $\alpha$-decay widths to ground band states in even-even nuclei. To this purpose we should first determine the deformation parameter $\delta$. We use the simplest ansatz of the CSM Hamiltonian, given by the harmonic term [26]

$$H_D(b_2) = A_1 \hat{N} = A_1 \sum_\mu b_{2\mu}^\dagger b_{2\mu} ,$$

(30)

depending upon the strength parameter $A_1$ and the deformation $\delta$. The one parameter description of the CSM Hamiltonian allows us to derive a universal relation for the energy ratio

$$\frac{E_{J+2}}{E_J} = \frac{\mathcal{I}_{J+2}(\delta) - \mathcal{I}_0(\delta)}{\mathcal{I}_{J}(\delta) - \mathcal{I}_0(\delta)} ,$$

(31)
in terms of the function (5), depending only on the deformation parameter \(d\).

![Figure 1](image_url)  

**Fig. 1** – Deformation parameter \(d\) versus the quadrupole deformation \(\beta_2\) [30].

We determined the optimal values of these parameters for each nucleus by using the fitting procedure for the energies of \(J = 2^+, 4^+, 6^+, 8^+\) ground band states. In our calculations we used 40 nuclei with known experimental values for \(\alpha\)-transitions to the first excited \(2^+\) state. In Fig. 1 we plotted the deformation parameter \(d\) as a function of the standard quadrupole deformation parameter \(\beta_2\) of Ref. [30]. This dependence can be fitted by two straight lines, corresponding to \(N > 126\) (filled circles) and \(N < 126\) (open circles).

In order to check the quality of this approach we plotted the relation (31) by solid lines for \(J = 2\) in Fig. 2 (a) and \(J=4\) in Fig. 2 (b). It is nicely fulfilled by all experimental data, with a higher accuracy for \(N > 126\) (filled circles) than for \(N < 126\) (open circles). This proves the predicting power of the CSM concerning the energies of the ground band from vibrational to well deformed nuclei.

We used the values of the deformation parameter \(d\) to estimate the effective charge by using the values \(B(E2: 2 \rightarrow 0)\) given by Eq. (8). The values are given in Fig. 3 by filled circles for \(N > 126\) and by open circles for \(N < 126\). They follow the linear dependence predicted by Eq. (9). The slope of the linear dependence determines the anharmonicity parameter \(a_q\).

The main purpose of this paper is to use the CSM deformation parameter \(d\), to determine the \(\alpha\)-decay fine structure, defined by the logarithm of the ratio between
Fig. 2 – (a) Theoretical ratio \( E_4/E_2 \) versus the deformation parameter \( d \) (solid line). By filled circles are given experimental values for the region \( N > 126 \) and by open circles for \( N < 126 \). (b) Same as in (a) but for the ratio \( E_6/E_4 \).

Fig. 3 – Effective charge (9) versus the deformation parameter \( d \). By filled circles are plotted the values for \( N > 126 \) and by open circles for \( N < 126 \). By solid lines are given the corresponding linear fits.

decay widths to ground and \( J^+ \) states, respectively

\[
I_J \equiv \log_{10} \left( \frac{\Gamma_0}{\Gamma_J} \right),
\]  

(32)
where partial widths are given by Eq. (29). We call this quantity the intensity of the $\alpha$-decay to the $J$-th state. In our calculations we fixed the value of the repulsive strength $c = 50$ MeV [17] and changed the only remaining free parameter $v_{0}$ in order to reproduce the Q-value for each $\alpha$-decay.

By changing the effective $\alpha$-daughter coupling strength $C(d)$, we reproduced the available experimental quadrupole intensities $I_{2}$. In Fig. 4 (a) we plotted the values of this strength as a function of the deformation $d$. They can be fitted by two linear dependencies. Thus, we indeed obtained the linear dependence of the $\alpha$-daughter QQ coupling strength with a negative slope, predicted by Eq. (22). Notice that for the region $N < 126$ the values belong to a narrow interval of small values $C \in [0.05, 0.1]$. It turns out that the QQ strength can also be related to the mass number $A$, as given in Fig. 4 (b). Let us mention that for $N > 126$ this behaviour is similar to the dependence of the reduced width squared (called $\alpha$-particle probability) for transitions connecting ground states with $J = 0$, defined as follows

$$\gamma_{J}^{2} = \frac{\Gamma_{J}}{2P_{J}},$$

where $P_{J}$ is the Coulomb penetrability computed at the touching radius [29]. Indeed, it turns out that the $\alpha$-daughter coupling strength is proportional to the $\alpha$-particle probability, but with different slopes for $N > 126$ (dark circles) and $N < 126$ (open symbols), as it can be seen from Fig. 5.

We used these values of the $\alpha$-daughter strength $C(d)$ reproducing $I_{2}$ values
Coherent state description of $\alpha$-transitions to excited states in even-even nuclei

in order to predict the intensities $I_4$ and $I_6$. They are given in Fig. 6 by dark symbols versus the index $n$. These values reproduce the available experimental data, plotted by open circles, with a reasonable accuracy. This proves that this relative simple model is able to simultaneously describe all available experimental $\alpha$-decay intensities to excited states within a range of four orders of magnitude.

4. CONCLUSIONS

We analysed the available experimental $\alpha$-decay widths to excited states by using the CSM formalism. We have shown that the simplest harmonic CSM Hamiltonian is able to describe all available energy level ratios in terms of the deformation parameter $d$, proportional to the standard quadrupole parameter $\beta_2$. We then described the $\alpha$-decay fine structure within the coupled channel formalism by using monopole plus quadrupole terms. We reproduced the $\alpha$-intensity to the $2^+$ state by using the QQ strength parameter. Its linear dependence on the deformation parameter confirms the CSM prediction. The QQ interaction strength has a jump of one order of magnitude by crossing the magic neutron number $N=126$. In the region above to $^{208}$Pb, i.e. for nuclei with largest $\alpha$-clustering components, it has the strongest value and diminishes again by one order of magnitude for $A=240$. By using these values we were able to reproduce experimental intensities to $4^+$ and $6^+$ states with a reasonable accuracy and we made theoretical predictions for the other $\alpha$-emitters.
Fig. 6 – (a) Predicted intensity $I_4$, by using the coupling strength $C(d)$ reproducing the corresponding value $I_2$, versus the index $n$ in the first column of the Table I. The labels correspond to the following isotope chains $n = 1 - 2 : Z = 76$, $n = 3 - 5 : Z = 78$, $n = 6 - 9 : Z = 84$, $n = 10 - 13 : Z = 86$, $n = 14 - 17 : Z = 88$, $n = 18 - 24 : Z = 90$, $n = 25 - 29 : Z = 92$, $n = 30 - 34 : Z = 94$, $n = 35 - 38 : Z = 96$, $n = 39 - 40 : Z = 98$. (b) Same as in (a), but for $I_6$.

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REFERENCES