The amplitude for double-beta decay with two-neutrino emission is related to $\beta^-$ and $\beta^+$ transitions of Fermi and Gamow-Teller type connecting ground state of initial and final nuclei with virtual intermediate nuclear states. The suppression of double Fermi and Gamow-Teller matrix elements has origin in violation of the isospin SU(2) and spin-isospin SU(4) symmetries, respectively. We study double Fermi matrix elements within an exactly solvable model. By using perturbation theory up to the first order a dependence of the two-neutrino double beta decay matrix element on the like-nucleon pairing, particle-particle and particle-hole proton-neutron interactions by assuming a weak violation of isospin symmetry of Hamiltonian expressed with generators of the SO(5) group. It is found that there is a dominance of transition through a single state of the intermediate nucleus.

Key words: two-neutrino double-beta decay, nuclear matrix element, SO(5) model.

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1. INTRODUCTION

Double beta decay has been a very interesting process as a probe examining the breakdown of total lepton number conservation due to finite mass of the Majorana neutrinos [1]. Double beta decay occurs in the case that a nucleus cannot undergo ordinary single beta decay because of energy conservation or angular momentum mismatch.

Much theoretical work has been done to calculate matrix elements and decay rates for both lepton number conserving two-neutrino double decay ($2\nu\beta\beta$-decay),

$$ (A, Z) \rightarrow (A, Z + 2) + 2e^- + 2\bar{\nu}_e, $$

and total lepton number non-conserving neutrinoless double beta decay,

$$ (A, Z) \rightarrow (A, Z + 2) + 2e^- . $$

The $2\nu\beta\beta$-decay, which involves the emission of two electrons and two antineutrinos, remains of major importance for nuclear physics. Till now, the $2\nu\beta\beta$-decay has been detected for 11 different nuclei for transition to the ground state and in two cases also to transition to $0^+$ excited state of the daughter nucleus. This rare
process is one of the major sources of background in running and planned experiments looking for a signal of the more fundamental neutrinoless double-beta decay, which occurs if the neutrino is a massive Majorana particle.

The $2\nu\beta\beta$-decay can be expressed in terms of sequential single beta decay Fermi and Gamow-Teller transitions through virtual intermediate states. The inverse half-life of the $2\nu\beta\beta$-decay can be written as

$$\left(T_{2\nu}^{1/2}\right)^{-1} = \frac{1}{2} \frac{G_{2}^{2} g_{A}^{4}}{M^{2}_{GT} - \left(\frac{g_{V}}{g_{A}}\right)^{2} M^{2}_{F}}$$

where $G_{2}$ is the lepton phase-space factor, $g_{A}$ ($g_{V}$) is the axial-vector (vector) coupling constant. The double Gamow-Teller (GT) and double Fermi (F) matrix elements are given by

$$M^{2}_{F;GT} = \sum_{n} \langle f \| \mathcal{O}_{F;GT} \| 1_{n}^{+} \rangle \langle 1_{n}^{+} \| \mathcal{O}_{F;GT} \| i \rangle \frac{E_{n} - (E_{i} + E_{f})/2}{E_{n}}$$

with

$$\mathcal{O}_{F} = \sum_{k=1}^{A} \tau_{k}^{+}, \quad \mathcal{O}_{GT} = \sum_{k=1}^{A} \tau_{k}^{+} \bar{\sigma}_{k}.$$
where

\[ N_i = \sum_m a_{m,t_i}^\dagger a_{m,t_i}, \quad \beta^- = \sum_m a_{m,-1/2}^\dagger a_{m,1/2}, \]

\[ S_i^\dagger = \frac{1}{2} \sum_m a_{m,t_i}^\dagger a_{m,t_i}, \quad P^- = \sum_m a_{m,-1/2}^\dagger a_{m,1/2}. \]

with \( i = p, n \) and \( t_{n,p} = \pm 1/2 \). \( a_{m,t}^\dagger \) (\( a_{m,t} \)) is creation (annihilation) operator of single particle state \( |jm,t\rangle \) for protons and neutrons \((t = t_p,t_n)\) and \( \tilde{a}_{m,t}^\dagger = (-1)^{j-m} a_{-m,t}^\dagger \).

We rewrite Hamiltonian (6) with help of operators

\[ A_y^\dagger(T_z) = \frac{1}{\sqrt{2}} \left[ a_y^\dagger \otimes a_y \right]_{T_z}, \quad N = N_p + N_n, \]

\[ T_z = \frac{N_n - N_p}{2}, \quad T^- = -\sqrt{2\Omega} \sum_m a_{m,-1/2}^\dagger a_{m,1/2}. \]

Here, \( A_y^\dagger(T_z) \) is the nucleon pair creation operator with angular momentum \( J = 0 \), isospin \( T = 1 \) and its projection on z-axis \( T_z \) \((T_z = 0, \pm 1)\). \( N, T_z \) and \( T^- \) are the particle-number operator, the isospin projection and the isospin lowering operators, respectively. It holds the identity \( T^2 = (T^-T^+ + T^+T^-)/2 + T_z^2 \). \( \Omega = j + 1/2 \) denotes the semi-degeneracy of the considered single level. The operators in the above relations with their Hermitian conjugates represent ten generators of the SO(5) group [4]. We assume, the system is in seniority \( s=0 \). Then, \([A_y^\dagger A_y]_0^0 \) expressed with the SO(5) Casimir operator [4] is given by

\[ [A_y^\dagger A_y]_0^0 = \frac{1}{2\sqrt{3\Omega}} \left[ (2\Omega + 3 - N/2)N/2 - T(T+1) \right]. \]

For the Hamiltonian (6) we get

\[ H = \left[ e_n + e_p - \frac{1}{3} \left( 3 + 2\Omega - \frac{N}{2} \right) \left( \frac{G_p + G_n}{2} + 2\kappa \right) \right] \frac{N}{2} \]

\[ + [e_n - e_p - 2\chi(T_z + 1)]T_z + \left[ 2\chi + \frac{1}{3} \left( \frac{G_p + G_n}{2} + 2\kappa \right) \right] T(T+1) \]

\[ + \frac{\Omega}{\sqrt{2}} \left( \frac{G_p - G_n}{2} \right) [A_y^\dagger A_y]_0^1 + \frac{\sqrt{2}}{3\Omega} \left( 4\kappa - \frac{G_p + G_n}{2} \right) [A_y^\dagger A_y]_0^2. \]

As a consequence of the presence of the isovector and isoquadrupole terms in Hamiltonian (10) the isospin is not conserved in general. It is due to differences between proton and neutron pairing strengths and an arbitrary strength of the proton-neutron isovector pairing component. However, particle number and isospin projection remains as good quantum numbers.

The kth eigenstates of the Hamiltonian (10) with quantum numbers \( N \) and \( T_z \) can be expressed in terms of a basis labelled by a chain of irreducible representations.
of the SO(5) group [4], namely

$$|k; NT_z\rangle = \sum_T c^{(k)}_{NT_T} |NTT_z\rangle.$$  \hspace{1cm} (11)

A diagonalization of $H$ requires calculation of matrix elements $\langle N, T, T_z | H | N, T, T_z \rangle$ and $\langle N, T \pm 2, T_z | H | N, T, T_z \rangle$. The corresponding reduced matrix elements are given in [2]. For $G_p = G_n$ and $(G_p + G_n)/2 = 4\kappa$ the Hamiltonian (10) is diagonal in the basis of states $|N, T, T_z\rangle$.

### 3. DOUBLE FERMI MATRIX ELEMENT WITHIN PERTURBATION THEORY

We shall assume a small violation of the isospin symmetry due to isotensor term of nuclear Hamiltonian (10). For the numerical example we consider a large value of $j$ to simulate the realistic situation corresponding to medium- and heavy-mass nuclei. The parameters chosen are given by

$$\Omega = 10, \quad N = 20, \quad 1 \leq T_z \leq 5,$$

$$e_p = 0.3 \text{ MeV}, \quad e_n = 0.1 \text{ MeV}, \quad G = 0.165 \text{ MeV},$$

$$G_p = G_n = G, \quad \chi = 0.044 \text{ MeV}, \quad 0.7 \leq 4\kappa/G \leq 1.3.$$ \hspace{1cm} (12)

For $4\kappa/G = 1$ the isospin symmetry is restored. In Fig. 1 we present $0^+$ states with energy $E_{TT_z}$ of different isotopes. This level scheme illustrates the situation for the $2\nu\beta\beta$-decay of $^{48}\text{Ca}$. The isospin is known to be, to a very good approximation, a valid quantum number in nuclei. The ground states of $^{48}\text{Ca}$ and $^{48}\text{Ti}$ can be identified with $T=4$ $T_z=4$ and $T=2$ $T_z=2$, respectively, i.e. they are assigned into different isospin multiplets. As the total isospin projection lowering operator $T^-$ is not changing the isospin the double Fermi matrix element $M^d_{F2}$ is non-zero only to the extent that the Coulomb interaction mixes the high-lying $T=4$ $T_z=2$ analogue of the $^{48}\text{Ca}$ ground state into the $T=2$ $T_z=2$ ground state of $^{48}\text{Ti}$.

We shall study double Fermi matrix element in the perturbation theory within the discussed model close to a point of a restoration of the isospin symmetry ($4\kappa/G = 1$). The isoscalar and isotensor terms of the Hamiltonian (10) represent the unperturbed and perturbed terms, respectively. We denote perturbed states and their energies with a superscript prime symbol ($|NTT_z\rangle$, $E_{TT_z}$) unlike the states with a definite isospin ($|NTT_z\rangle$, $E_{TT_z}$). Up to the first order of parameter $(4\kappa - G)$ we find

$$E'_{44} = 14e_n + 6e_p - \frac{110}{3} (G + 2\kappa) - \sqrt{\frac{2}{3}} \Omega (G - 4\kappa) \langle 44 | [A^1 \tilde{A}]_0^2 | 44 \rangle,$$

$$E'_{43} = 13e_n + 7e_p + 16\chi - \frac{110}{3} (G + 2\kappa) - \sqrt{\frac{2}{3}} \Omega (G - 4\kappa) \langle 43 | [A^1 \tilde{A}]_0^2 | 43 \rangle,$$

$$E'_{22} = 12e_n + 8e_p - \frac{124}{3} (G + 2\kappa) - \sqrt{\frac{2}{3}} \Omega (G - 4\kappa) \langle 22 | [A^1 \tilde{A}]_0^2 | 22 \rangle.$$ \hspace{1cm} (13)
Fig. 1 – The energy of the $0^+$ states of different isotopes are shown for $j = 19/2$ (and the set of parameters (12) with $4\kappa/G = 1$) in MeV vs. $Z$ plot. States are labeled by $(T, T_z)$.

The particular matrix elements of SO(5) operators connecting states with a definite isospin and its projection are presented in [2]).

The double Fermi matrix element can be written as

$$M_{F}^{2\nu} = \sum_{T=4,6,8,10}^{10} \frac{\langle 2'2| T^- |T'3\rangle \langle T'3| T^- |4'4\rangle}{E'_{T3} - (E'_{44} - E'_{22})/2}. \quad (14)$$

It contains a sum over the states of the intermediate nucleus $|T'3\rangle$. However, up to second order of perturbation theory there is only a single contribution through the intermediate state $|4'3\rangle$. Thus, we have

$$M_{F}^{2\nu} \sim \frac{\langle 2'2| T^- |4'3\rangle \langle 4'3| T^- |4'4\rangle}{E_{43} - (E'_{44} - E'_{22})/2}. \quad (15)$$

If isospin symmetry is restored ($4\kappa = G$) we end up with $\langle 2'2| T^- |4'3\rangle = \langle 22| T^- |43\rangle = 0$. For the energy denominator in (15) with help of Eqs. (13), we get

$$E'_{43} - (E'_{44} - E'_{22})/2 = 16\chi + \frac{7}{3} (G + 2\kappa)$$

$$+ \sqrt{\frac{T}{6}} \Omega (4\kappa - G) \left[ 2 \langle 43| [A^1 \bar{A}]_0^0 |43\rangle - \langle 44| [A^1 \bar{A}]_0^0 |44\rangle - \langle 22| [A^1 \bar{A}]_0^0 |22\rangle \right] \quad (16)$$

We note that the energy denominator $E'_{43} - (E'_{44} - E'_{22})/2$ as well as the whole
double Fermi matrix element $M_{F}^{2\nu}$ does not depend explicitly on mean field parameters $\epsilon_{p}$ and $\epsilon_{n}$. If we restrict our consideration to the first order perturbation theory we find

$$M_{F}^{2\nu} \approx \frac{\langle 42 | T^{-} | 43 \rangle \langle 43 | T^{-} | 44 \rangle}{16 \chi + \frac{1}{3} (G + 2 \kappa)} \times \sqrt{\frac{2}{3} \Omega (G - 4 \kappa)} \frac{\langle 42 | [A^{1}_{\mu} | 22 \rangle}{(28 \chi + 14/3 (G + 2 \kappa))}.$$ 

![Graph](image)

Fig. 2 – (Color online) Matrix element $M_{F}^{2\nu}$ for the double-Fermi two-neutrino double-beta decay mode as function of ratio $4\kappa/G$ for a set of parameters (12). Exact results are indicated with a solid line. The results obtained within the perturbation theory up to the first order in isotensor contribution to Hamiltonian are shown with dashed line, respectively. The restoration of isospin symmetry is achieved for $4\kappa/G = 1$.

In Fig. 2 $M_{F}^{2\nu}$ is plotted as function of ratio $4\kappa/G$. We see that double Fermi matrix element calculated with perturbation theory up to the first order reproduces well the exact solution close to the point of restoration of isospin symmetry ($4\kappa/G = 1$).

4. CONCLUSIONS

The $2\nu\beta\beta$-decay processes of the Fermi type is analysed in the context of violation of isospin SU(2) symmetry. Good isospin forbids the $2\nu\beta\beta$-decay as ground
states of initial and final nuclei belong to different isospin multiplets. One needs an isotensor force to mix $\Delta T = 2$. Naturally, the Coulomb interaction contains such an isotensor force. In our case we break isospin symmetry by hand. The only isospin violation comes from the difference of the proton-proton ($G_p$) and the neutron-neutron ($G_n$) pairing force compared to the proton-neutron isospin $= 1$ pairing force ($\kappa$).

By taking the advantage of perturbation theory double Fermi matrix elements is analysed in exactly solvable model, which Hamiltonian is expressed with generators of the SO(5) group. Up to the first order of perturbation theory in the isotensor contribution to the Hamiltonian explicit dependence of $M^2_F$ on different components of residual interaction of nuclear Hamiltonian, namely like-nucleon pairing, particle-particle and particle hole proton-neutron interactions, is found. The mean-field part of Hamiltonian does not enter explicitly in this decomposition of double Fermi matrix element and is related only to the calculation of unperturbated states of Hamiltonian. Further, a dominance of a contribution through a single state of the intermediate nucleus to $M^2_F$ is established.

It goes without saying that more comprehensive calculation of double Fermi matrix element up to the second order in perturbation theory is desired. There is also a possibility to study double Gamow-Teller matrix element in the context of the violation of SU(4) symmetry [6] in the perturbation theory as well. First, this can be achieved within the SO(8) model. There is also a possibility to exploit perturbation theory also for the case of realistic calculation of two-neutrino double beta decay matrix elements.

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REFERENCES