ON THE QUANTUM-CLASSICAL ANALOGIES

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The issue of analogies between classical optics and quantum mechanics is analyzed in details. The appropriateness and limitations of these analogies are discussed for both Schrödinger and Dirac type quantum states.

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1. INTRODUCTION

It is with great pleasure that I contribute to the special issue of Romanian Journal of Physics, dedicated to the 70th anniversary of Prof. Apolodor Raduta. I have known Prof. Raduta with the occasion of a research project submission. Although the project was unsuccessful, it was a great opportunity to meet some extraordinary people, among them Prof. Raduta.

In this issue I would like to address briefly one of the research issues that is constantly present throughout my activity: the analogies between the classical and the quantum worlds. I became interested in this subject when I started to use the Wigner distribution function (WDF) [1] as a mathematical tool suitable for describing optical sources and systems. At that moment, I was particularly intrigued by the affirmation that [2] negative values of the WDF are a signature of the quantum nature of the system under investigation. I knew that this statement was wrong because negative values of WDF have been experimentally generated in classical optics [3-4]. On the other hand, it was quite difficult to convince quantum physicists of this reality. The reason is at the same time simple and subtle. It is usually assumed that quantum mechanics is a totally different theory (conceptually as well as formally) than classical mechanics. Indeed, a quantum particle cannot be represented by a point in phase space, for example, but by an area that cannot be less than $h$ due to the uncertainty principle. However, there are classical theories in...
physics that share this nonlocal character with quantum mechanics: wave theories, in particular, optics. An optical beam cannot either be represented by a point in phase space [5]; only light rays can be represented as such, but light rays are approximations of electromagnetic waves, in a similar way in which classical mechanics is an approximation of quantum mechanics.

In this paper I would like to comment on the analogies between quantum mechanics and classical optics, starting from the definitions of analogy found in the Oxford dictionary: “a comparison between one thing and another, typically for the purpose of explanation or clarification”, [6] and in Wikipedia: “Analogy […] is a cognitive process of transferring information or meaning from a particular subject (the analogue or source) to another particular subject (the target)” [7].

2. APPROPRIATENESS AND LIMITATIONS OF CLASSICAL-QUANTUM ANALOGIES

In view of the latter definitions, classical optics, or at least some appropriately chosen concepts, can aid us in understanding specific quantum concepts or theories. But how in-depth can we use such analogies? Are there quantum concepts/systems with no classical analogs? Are, actually, these analogies appropriate/useful? In fact, it is not at all obvious that analogies could exist between classical optics and quantum mechanics. I would like to recall that quantum mechanical Hamiltonians of are commonly introduced by replacing with operators the dynamical variables in the classical mechanical Hamiltonian

\[
H(r, p) = \frac{p^2}{2m(r)} - V(r)
\]

(1)

where \( m \) and \( V \) are the (effective) mass and potential energy. On the contrary, in classical optics the Hamiltonian \( H(r, p) = -\{n^2(r) - p^2\}^{1/2} \) is defined starting from the Fermat principle and [8], in the paraxial approximation only, it acquires a form similar to (1):

\[
H(r, p) = \frac{p^2}{2n(r)} - n(r)
\]

(2)

where \( n \) is the refractive index. The structure of the optical and classical mechanical Hamiltonians are different mathematically: a restriction similar to \( |p| \leq n \) in optics does not exist in classical mechanical and, moreover, the formal equivalent parameters to the uncorrelated (effective) mass and potential energy in classical mechanics become identical in optics. Yet, it is possible to find analogies between the propagation of classical optical beams (constituted, from a quantum mechanical point of view, from massless photons) and the evolution of massive quantum particles.
In fact, the analogies between classical optics governed by the Maxwell equations and massive particles described by the Schrödinger equation are known for quite a long time [9] and have inspired the development of research areas such as particle optics [10] or photonic crystals [11]. Such analogies can only exist for time-independent systems and they do not rely on the Hamiltonians (1) and (2) but on the formal similarity between the Helmholtz equation obeyed by any component $A$ of the monochromatic electromagnetic field with frequency $\omega$,

$$\nabla^2 A + (\omega n / c)^2 A = 0 \tag{3}$$

where $c$ is the speed of light, and the time-independent Schrödinger equation for the wavefunction $\Psi$:

$$\nabla^2 \Psi + [2m(E - V)/\hbar^2] \Psi = 0 \tag{4}$$

Indeed, (3) and (4) resemble each other, although $A$ is a vector and $\Psi$ a scalar function.

What happened with the differences in (1) and (2) between classical optics and classical/quantum mechanics? One of the origins of the differences between classical optics and classical mechanics is that the Hamiltonian formulation in optics is associated with the phase/the eikonal function/the wavefront of light waves. As such, it is related to a non-local description of electromagnetic waves. By replacing $p$ and $r$ in the classical Hamiltonian (1) with their corresponding operators, quantum mechanics becomes itself a non-local theory. In addition, using (3) instead of (2) to describe the evolution of optical beams, not only the phase but also the amplitude of electromagnetic fields is taken into account, as in (4). At this point, a simple equivalence between $\Psi$ and a component of $A$ and a corresponding replacement of the optical propagation constant/wavenumber $k = \omega n / c$ with the quantum one $\gamma = \sqrt{2m(E - V)/\hbar}$ almost lead to a quantitative equivalence between the propagation of ballistic quantum wavefunctions and of optical fields. “Almost” because boundary conditions need also to be similar. Because the boundary conditions at interfaces are different for transverse electric and transverse magnetic polarized radiation, a set of analogous optical/quantum parameters can be found separately for the two cases [12]. Now, one can easily design optical structures with the same transmittance and reflectance as the transmission and reflection probabilities of quantum wavefunctions propagating through a succession of layers with different effective electron masses and/or potential energies. We have a quantitative optical-quantum analogy.

Of course, the fact that optical fields and quantum wavefunction evolve in similar ways does not mean that their meaning is the same. The advantage is that optical amplitude and phase measurements could give us a hint on the quantum wavefunction, which are much more difficult to measure. In addition, although the analogies between (3) and (4) are strictly valid only for ballistic (without
collisions) quantum propagation, they can be extended to multiple scattering regimes as long as the phase coherence is not completely lost [13]. Random lasers [14], for example, are a result of pursuing such analogies with strongly localized electrons in random nanostructures.

Are we to declare then that classical optics is not so much different from quantum mechanics? Of course, the answer is no. But why? We have to look now to an essentially quantum property: entanglement. It is usually considered that entanglement has no classical analog. Is it true? Not exactly. Entanglement exists also in classical optics (or any classical wave theory) in a restrictive sense, i.e. as non-factorizability of a state vector. More precisely, we can have local entanglement between different degrees of freedom (momentum/direction and polarization, for example) of a single light beam [15-16]. In particular, there are optical analogs of quantum bits (qubits) in the form of optical cebits [16], defined as superpositions of classical beams, and there are even implementations of quantum algorithms by classical optical systems [17-18]. However, nonlocal correlations/multiparticle entanglement between spatially separated states cannot be mimicked in classical optics. This is the reason why the scaling behavior of qubits (the exponential decrease of computation time with a linear increase in the input states) differs from that of cebits. A number of \( n \) optical components/beams do not lead to \( 2^n \) optical paths [15].

Is this the end of the story? Is quantum mechanics only different from classical optics through nonlocal entanglement? Not quite. I have mentioned in the beginning of this short paper that the starting point of interest in the classical-quantum analogies was the phase space formalism of classical optics. This formalism is particularly suited for comparing quantum and classical optics theories because it uses the same mathematical language in both cases (the opposite approach, that of expressing classical optical beams and systems via quantum mechanical operators can also be successfully applied [19-20]).

In a, let’s say, two-dimensional phase space \((x, p)\), time-independent quantum mechanical systems must be represented not by localized points, due to the uncertainty principle but by quasiprobability distributions \(F(x, p)\) that lead to correct forms of the observable quantities and similar expectation values of arbitrary operators \(\hat{A}(\hat{x}, \hat{p})\) as in classical phase spaces [21], i.e.

\[
\langle \hat{A}(\hat{x}, \hat{p}) \rangle = \int A(x, p)F(x, p)dx dp,
\]

where \(A(x, p)\) is the function obtained from \(\hat{A}(\hat{x}, \hat{p})\) by replacing the position and momentum operators \(\hat{x}, \hat{p}\) with scalar variables \(x\) and \(p\), respectively. If the Weyl or symmetric rule of association is obeyed at these replacements the quasiprobability function is the WDF. It is defined for pure states as
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\[ W(x, p, t) = \frac{1}{2\pi\hbar} \int \exp(-ipq/h)\Psi^*\left(x - \frac{q}{2}, t\right)\Psi\left(x + \frac{q}{2}, t\right) dq, \]  

(6)

and obeys a series of useful properties (it has real values, satisfies the marginal properties and the Moyal formula, transforms in a simple way at spatial and angular translations, etc. [1, 21]). A phase space treatment of quantum mechanics is equivalent to the Schrödinger, Heisenberg or interaction representations and can be expressed also by a series of axioms [22].

I would like to emphasize here only that a classical optical WDF can be similarly defined in terms of any component of the electromagnetic field except that the normalized Planck constant is replaced by the normalized wavelength of light (see the reviews in [5, 23]).

The interest in the phase space treatment of classical optics and quantum mechanics is motivated by the fact that all previously discussed analogies are retrieved also in phase space since the WDF is defined in terms of wavefunctions/electromagnetic field components. In particular, the modes of graded index optical waveguides are the analogs of Fock states [24-25], while the quantum squeezing operator is implemented in classical optics by systems consisting of lenses and free spaces [25]. However, since the WDF is bilinear in the wavefunction/electromagnetic field component, it displays interference terms whenever a superposition of quantum or classical states appear, irrespective of the classical or quantum nature of states’ interference; we can have phase space interference terms even if the states do not actually interfere in the configuration space. In particular, although quantum interference between spatially separated states cannot be mimicked by classical means, the WDF of a superposition of two coherent classical fields with different spatial or angular positions has the same form as that of a quantum Schrödinger cat state [26]; the measured interference term has positive and negative values, as in the quantum case [4]. In this sense the quantum-classical analogies in phase space seem to reach a deeper level than in configuration space. There is actually no difference at all in the phase space treatment of quantum and classical worlds, the phase space approach, and in particular the WDF, being an ideal mathematical tool in a unified formalism [22, 27-28]; using this mathematical language a quantum system need no longer be separated from the classical environment. As such, the question of appropriateness and limits of quantum-classical analogies must be answered very carefully.

3. CLASSICAL ANALOGIES OF QUANTUM MASSLESS PARTICLES

In the previous section we have discussed the classical optical analogies of quantum massive particles described by the Schrödinger equation. However, the recent surge in interest related to graphene [29] has put forward the issue of
classical optical analogies of quantum states described by a massless Dirac-like equation. At first it seemed that such analogies could be found easier since the photons are also massless particles. But such considerations proved themselves to be naïve. In fact it is much harder to find classical optical analogs to quantum states in graphene and the corresponding optical systems are not common. Because the wavefunction in graphene is spinorial, $\Psi = (\psi_1, \psi_2)^T$, classical optical analogies have been searched for in the form of polarized states of light propagating in gyrotropic and electro-optic materials [30-31] or in the form of electromagnetic fields evolving through common [32] or complex conjugate media [33]. In all cases quantitative analogies could be found, in the sense that the optical transmittance and reflectance through specific structures was identical to the transmission and reflection probabilities in graphene-based structures. However, it is harder to follow these analogies than in the case of the corresponding similarities between classical optics and Schrödinger particles. In particular, it was not possible to define a set of analogous classical-quantum parameters because the form of the optical and quantum reflection and transmission coefficients was different in all cases; only the square modulus of these coefficients could be equated. The reason of the difficulty of finding simple classical optical analogs for graphene lies in the chiral nature of quantum states of graphene. Chirality has no classical analog; it can only be mimicked by electromagnetic waves in very special materials. With one notable exception: photonic structures with Dirac points [34-35]. In this case the two spinorial components of the wavefunction in graphene are replaced by two degenerate optical Bloch states that satisfy, for certain frequencies of the incident radiation, the same massless Dirac-like equation. The quantum-classical analogies become again exact since chirality is strongly related to the honeycomb lattice, but the classical and quantum fields retain their conceptual differences. Only the mathematical formalism is identical, as are, for example, the reflection and transmission coefficients through certain lengths of graphene-like honeycomb lattices.

4. CONCLUSIONS

After this brief discussion on the quantum-classical optical analogies, I believe that the main conclusion is that this problem has different levels of understanding and answers. There are quantum systems that have classical analogs in phase space only, there are quantum states described by the Schrödinger equation that propagate through specific structures in exactly the same way as electromagnetic fields through appropriately designed optical structures, and finally there are quantum states described by a Dirac-like equation that are analogous to optical fields propagating through special materials or through specifically designed honeycomb lattices. Is the quantum-classical analogy useful? There is no doubt about its usefulness taking into account the feedback it produced.
between different areas of physics. In addition, the quantum-classical analogies help us understand better the fundamental differences between classical and quantum theories. Is the analogy appropriate? It is our task to find appropriate quantum and classical systems that are analogous; they probably exist. What are the limits of the analogy? Again, it depends on our skills to establish or surpass the limits of classical analogies to quantum systems. All such analogies, if they exist, are based on the non-local character of quantum and optical states. These analogies must be used with care, in the sense that the conceptual differences between the quantum and classical worlds do not disappear if an analogy is established. The art of analogies in physics is to compare different systems for the purpose of clarification (as in the definition in [6]) rather than to transfer meaning from a particular research field to another (as in the definition in [7]). Of course, a transfer of information from one research field to another (as implied by the definition in [7]) is always welcome.

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