In the framework of the theory of open systems based on completely positive quantum dynamical semigroups, we give a description of the continuous variable entanglement for a system consisting of two non-interacting bosonic modes embedded in a thermal environment. By using the entanglement of formation, it is described the evolution of entanglement in terms of the covariance matrix for symmetric Gaussian input states. In the case of an entangled initial squeezed vacuum state, entanglement suppression (entanglement sudden death) takes place, for all temperatures of the thermal bath. For definite values of temperature and dissipation constant, one can observe temporary revivals of the entanglement, but for long times the system evolves to an equilibrium state which is always separable.

Key words: Quantum entanglement, open systems, Gaussian states.

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I am very pleased to have written this paper in the Honour of Professor Apolodor Aristotel Răduţă and on the occasion of his 70th Anniversary, to wish him a long life in good health and further success in his scientific activity.

1. INTRODUCTION

In recent years there is an increasing interest in using continuous variable systems in applications of quantum information processing, communication and computation [1]. In the special case of Gaussian states there exist necessary and sufficient criteria of entanglement [2, 3] and quantitative entanglement measures [4, 5]. In quantum information theory of continuous variable systems, Gaussian states, in particular two-mode Gaussian states, play an important role since they can be easily created and controlled experimentally. Implementation of quantum communication and computation encounters the difficulty that any realistic quantum system cannot be isolated and it always has to interact with its environment. Decoherence and dynamics of quantum entanglement in continuous variable open systems have been intensively studied in the last years [6-18]. The Markovian time evolution of quantum
correlations of entangled two-mode continuous variable states has been examined in single-reservoir [19] and two-reservoir models [20,21], representing noisy correlated or uncorrelated Markovian quantum channels.

In a series of papers we studied, in the framework of the theory of open systems based on completely positive quantum dynamical semigroups, the dynamics of the entanglement of two bosonic modes coupled to a common thermal environment [22-26], and also in the case when each one of the two modes is coupled to its own thermal reservoir [27], by using the logarithmic negativity as a quantitative measure of the entanglement. In the present work we study the dynamics of the entanglement of two modes coupled to a common thermal reservoir, by using the entanglement of formation (EoF) as a quantitative measure of the entanglement. We are interested in discussing the correlation effect of the environment, therefore we assume that the two modes are uncoupled, i.e. they do not interact directly. The initial state of the subsystem is taken of Gaussian form and the evolution under the quantum dynamical semigroup assures the preservation in time of the Gaussian form of the state.

The paper is organized as follows. In Sec. 2 we write the Markovian master equation in the Heisenberg representation for an open system interacting with a general environment and the evolution equation for the covariance matrix. As a solution of this equation we give the time evolution of the covariance matrix for an initial Gaussian state. By using the EoF for two-mode symmetric Gaussian states [5], we investigate in Sec. 3 the dynamics of entanglement for the considered open system. In particular, using the asymptotic covariance matrix, we determine the behaviour of the entanglement in the limit of long times. We show that in the case of an entangled initial squeezed vacuum state, entanglement suppression takes place, for all temperatures of the reservoir. For definite values of temperature and dissipation constant, one can observe temporary revivals of the entanglement, but in the limit of large times the system evolves to an equilibrium state which is always separable. A summary is given in Sec. 4.

2. MASTER EQUATION FOR OPEN QUANTUM SYSTEMS

We study the dynamics of a system composed of two identical non-interacting bosonic modes in weak interaction with a thermal environment. In the axiomatic formalism based on completely positive quantum dynamical semigroups, the Markovian irreversible time evolution of an open system is described by the following quantum Markovian Kossakowski-Lindblad master equation for an operator $A$ in the Heisenberg representation ($\dagger$ denotes Hermitian conjugation) [28,29]:

$$\frac{dA(t)}{dt} = i\hbar[H, A(t)] + \frac{1}{2\hbar} \sum_j (V_j^\dagger [A(t), V_j] + [V_j^\dagger, A(t)] V_j).$$ (1)
Here, $H$ denotes the Hamiltonian of the open system and the operators $V_j, V_j^\dagger$, defined on the Hilbert space of $H$, represent the interaction of the open system with the environment. We are interested in the set of Gaussian states, therefore we introduce such quantum dynamical semigroups that preserve this set during time evolution of the system.

The Hamiltonian of the two uncoupled resonant harmonic oscillators of identical mass $m$ and frequency $\omega$ is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m\omega^2}{2}(x^2 + y^2),$$  \hspace{1cm} (2)

where $x, y$ are the coordinates and $p_x, p_y$ are the momenta of the two bosonic modes.

The equations of motion for the quantum correlations of the canonical observables are the following ($T$ denotes the transposed matrix) [30]:

$$\frac{d\sigma(t)}{dt} = Y\sigma(t) + \sigma(t)Y^T + 2D,$$  \hspace{1cm} (3)

where

$$Y = \begin{pmatrix} -\lambda & 1/m & 0 & 0 \\ -m\omega^2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1/m \\ 0 & 0 & -m\omega^2 & -\lambda \end{pmatrix}, \quad D = \begin{pmatrix} D_{xx} & D_{xp_x} & D_{xy} & D_{xp_y} \\ D_{xp_x} & D_{pp_x} & D_{yp_y} & D_{pp_y} \\ D_{xy} & D_{yp_y} & D_{yy} & D_{yp_y} \\ D_{xp_y} & D_{pp_y} & D_{yp_y} & D_{pp_y} \end{pmatrix},$$  \hspace{1cm} (4)

and the diffusion coefficients $D_{xx}, D_{xp_x}, ...$ and the dissipation constant $\lambda$ are real quantities. We introduced the following $4 \times 4$ bimodal covariance matrix:

$$\sigma(t) = \begin{pmatrix} \sigma_{xx}(t) & \sigma_{xp_x}(t) & \sigma_{xy}(t) & \sigma_{xp_y}(t) \\ \sigma_{xp_x}(t) & \sigma_{pp_x}(t) & \sigma_{yp_y}(t) & \sigma_{pp_y}(t) \\ \sigma_{xy}(t) & \sigma_{yp_y}(t) & \sigma_{yy}(t) & \sigma_{yp_y}(t) \\ \sigma_{xp_y}(t) & \sigma_{pp_y}(t) & \sigma_{yp_y}(t) & \sigma_{pp_y}(t) \end{pmatrix},$$  \hspace{1cm} (5)

where its elements are defined as $\sigma_{ij} = \langle R_i R_j + R_j R_i \rangle / 2, i, j = 1, ..., 4$, with $R = \{x, p_x, y, p_y\}$, which up to local displacements fully characterize any Gaussian state of a bipartite system.

The time-dependent solution of Eq. (3) is given by [30]

$$\sigma(t) = M(t)[\sigma(0) - \sigma(\infty)]M^T(t) + \sigma(\infty),$$  \hspace{1cm} (6)

where the matrix $M(t) = \exp(Yt)$ has to fulfill the condition $\lim_{t \to \infty} M(t) = 0$. The values at infinity are obtained from the equation

$$Y\sigma(\infty) + \sigma(\infty)Y^T = -2D.$$  \hspace{1cm} (7)
3. ENTANGLEMENT OF FORMATION FOR SYMMETRIC GAUSSIAN STATES

One of the main tasks of quantum information theory is to quantify the entanglement and the quantum correlations of quantum states. For this purpose, several entanglement measures have been introduced. In particular, two of such measures – the entanglement of formation and logarithmic negativity – have a well defined physical meaning. Given an entangled state, the EoF for this state expresses the number of maximally entangled states one needs to create this state [31]. The formal definition of the EoF is

$$E_f(\rho) = \inf \sum_j p_j E(\psi_j),$$

where $E(\psi_j)$ is the von Neumann entropy of the pure state $\psi_j$ and we take the infimum over all pure-state decompositions of $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, $\sum_j p_j = 1$. For symmetric Gaussian states an analytical expression for the EoF was obtained in [5, 32].

A Gaussian state has a Gaussian Wigner function in phase space and it is completely characterized by its first and second moments of canonical variables. Namely, a two-mode Gaussian state is entirely specified by its covariance matrix (5), which is a real, symmetric and positive matrix with the following block structure:

$$\sigma(t) = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

where $A$, $B$ and $C$ are $2 \times 2$ Hermitian matrices. $A$ and $B$ denote the symmetric covariance matrices for the individual reduced one-mode states, while the matrix $C$ contains the cross-correlations between modes. The covariance matrix (9) (where all first moments have been set to zero by means of local unitary operations which do not affect the entanglement) contains four local symplectic invariants in form of the determinants of the block matrices $A, B, C$ and covariance matrix $\sigma$. We will concentrate here on symmetric states, i.e. those with $\det A = \det B$.

Based on these invariants, the following expression of the EoF was obtained for bipartite symmetric Gaussian states [5, 32]:

$$E_f(\rho) = g(\Delta); \Delta = 2\sqrt{\det A - \det C - \sqrt{\text{Tr}(AJCJBC^TJ) - 2\det A\det C}},$$

where $J$ is the $2 \times 2$ symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$g(\Delta) = c_+(\Delta) \log_2[c_+(\Delta)] - c_-(\Delta) \log_2[c_-(\Delta)], \quad c_{\pm} = (\Delta^{1/2} \pm \Delta^{1/2})^2/4.$$
The covariance matrix describes an entangled state if and only if $\Delta < 1$, and for $\Delta \geq 1$ the state is separable.

In this work we evaluate the EoF for the symmetric Gaussian states of two bosonic modes, interacting with a thermal environment. We suppose that the asymptotic state of the considered open system is a Gibbs state corresponding to two independent bosonic modes in thermal equilibrium at temperature $T$. Then the quantum diffusion coefficients have the following form (we put from now on $\hbar = 1$): [29]

$$
\begin{align*}
    m\omega D_{xx} &= \frac{D_{pp_x}}{m\omega} = \frac{\lambda}{2} \coth \frac{\omega}{2kT}, \\
    m\omega D_{yy} &= \frac{D_{pp_y}}{m\omega} = \frac{\lambda}{2} \coth \frac{\omega}{2kT}, \\
    D_{xp} &= D_{yp} = 0.
\end{align*}
$$

(13)

In the following, we analyze the dependence of the EoF $E_f(t)$ on time $t$ and temperature $T$ of the thermal reservoir, when the diffusion coefficients are given by Eqs. (13). We consider entangled initial Gaussian states.

The evolution of an entangled initial state is illustrated in Figs. 1, 2, where we represent the dependence of the functions $E_f(t)$ and $\Delta(t)$ on time $t$ and temperature $RJP 58(Nos. 9-10), 1355–1362 (2013) (c) 2013-2013$
for an entangled initial squeezed vacuum state of the form

$$\sigma(0) = \frac{1}{2} \begin{pmatrix}
\cosh 2r & 0 & \sinh 2r & 0 \\
0 & \cosh 2r & 0 & -\sinh 2r \\
\sinh 2r & 0 & \cosh 2r & 0 \\
0 & -\sinh 2r & 0 & \cosh 2r
\end{pmatrix},$$

(14)

where \( r \) denotes the squeezing parameter. We observe that for a given temperature \( T \), at certain finite moment of time, \( \Delta(t) \) becomes larger or equal to 1 and therefore the state becomes separable. This is the so-called phenomenon of entanglement sudden death. This behaviour is in contrast to that one of the quantum decoherence, during which the loss of quantum coherence is usually gradual [14, 33]. For the moments when the state is entangled, \( E_f(t) \) quantifies the corresponding degree of entanglement. For definite values of temperature \( T \) and dissipation constant \( \lambda \), one can observe temporary revivals of the entanglement, but in the limit of large times the system evolves to an equilibrium state which is always separable.

The dynamics of entanglement of the two modes strongly depends on the initial states and the coefficients describing the interaction of the system with the thermal reservoir (temperature and dissipation constant).
On general grounds, one expects that the effects of decoherence, counteracting entanglement production, is dominant in the long-time regime, so that no quantum correlations (entanglement) is expected to be left at infinity. Indeed, using the diffusion coefficients given by Eqs. (13), we obtain from Eq. (7) the following elements of the asymptotic matrices $A(\infty)$ and $B(\infty)$:

$$m\omega\sigma_{xx}(\infty) = \frac{1}{m\omega} \sigma_{xx}(\infty) = \frac{1}{2} \coth \frac{\omega}{2kT}, \quad \sigma_{pp}(\infty) = 0,$$

$$m\omega\sigma_{yy}(\infty) = \frac{1}{m\omega} \sigma_{yy}(\infty) = \frac{1}{2} \coth \frac{\omega}{2kT}, \quad \sigma_{pp}(\infty) = 0,$$

while all the elements of the entanglement matrix $C(\infty)$ are zero. Then $\Delta(t)$ (10) takes the following form in the limit of large times:

$$\Delta(\infty) = \coth \frac{\omega}{2kT} \geq 1$$

and, consequently, the equilibrium asymptotic state is always separable in the case of two identical non-interacting bosonic modes immersed in a thermal reservoir.

In Refs. [34-38] we described the time evolution of the logarithmic negativity for two bosonic modes interacting with a common thermal bath. For large times it depends on temperature only, and does not depend on the initial Gaussian state. Our present results confirm the previously obtained ones, according to which the asymptotic state is always separable.

### 4. SUMMARY

In the framework of the theory of open systems based on completely positive quantum dynamical semigroups, we investigated the Markovian dynamics of the quantum entanglement for a subsystem composed of two non-interacting modes embedded in a common thermal bath. We have described the evolution of entanglement in terms of the covariance matrix for symmetric Gaussian input states and we have determined the corresponding entanglement of formation. In the case of an entangled initial Gaussian state (squeezed vacuum state), entanglement suppression (entanglement sudden death) takes place. For definite values of temperature and dissipation constant, one can observe temporary revivals of the entanglement, but the system evolves in the limit of large times to an equilibrium state which is always separable.

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