RECURRENCE RELATIONS FOR THE NUMBER OF SOLUTIONS OF A CLASS OF DIOPHANTINE EQUATIONS

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Recursive formulas are derived for the number of solutions of linear and quadratic Diophantine equations with positive coefficients. This result is further extended to general non-linear additive Diophantine equations. It is shown that all three types of the recursion admit an explicit solution in the form of complete Bell polynomial, depending on the coefficients of the power series expansion of the generating functions for the sequences of individual terms in the Diophantine equations.

Key words: Diophantine equations, the number of solutions, recursion, generating function, Bell polynomials.

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1. INTRODUCTION

Diophantine equations are encountered in theory of partitions, combinatorial analysis, integer linear programming, and in many related areas [1, 2]. Although not as common, Diophantine equations occur in physical applications. We mention the problems in solid state physics [3–6] and theory of angular momentum [7, 8]. A new application area emerged recently in the field of nuclear physics in connection with the problem of calculating degeneracy of the symmetry group reduction chains [9].

In this paper the problem of estimating the number of solutions of Diophantine equations is discussed from the viewpoint of Ward identities, successfully used earlier to establish relationships between different kinds of invariant integrals on continuous groups (see, e.g., [10]). In Sects. 2 and 3, we consider linear and quadratic Diophantine equations and derive recursive formulas for the number of solutions. In Sect. 4 we show that the number of solutions of general non-linear additive Diophantine equations can also be calculated recursively, which involves the partial Bell polynomials evaluated at the first expansion coefficients of the generating functions for the sequences of individual terms in the equation. In conclusion of Sect. 4 we show, furthermore, that the number of solutions is equal to the complete Bell polynomial evaluated at the first expansion coefficients of the logarithm of the full generating function.

2. LINEAR DIOPHANTINE EQUATION

We consider the linear Diophantine equation
\[ a_1 k_1 + \ldots + a_r k_r = n \]  
with positive integer coefficients \( a_l \in \mathbb{N} \) \((l = 1, \ldots, r)\) and an integer \( n \). We are looking for the number \( \nu_r(n) \) of non-negative integer solutions \( k_l \in \mathbb{N}_0 \) \((l = 1, \ldots, r)\) of this equation.

**Theorem 1** The number of non-negative solutions of the linear Diophantine equation (1) with positive integer coefficients \( a_l \) can be obtained by the following recursive formula:
\[ \nu_r(n) = \frac{1}{n} \sum_{l=1}^{r} a_l \left( \sum_{i=1}^{\lfloor n/a_l \rfloor} \nu_r(n - ia_l) \right), \]  
with the initial conditions \( \nu_r(n) = 0 \) for \( n < 0 \) and \( \nu_r(0) = 1 \).

**Proof** The number of solutions can be found by enumerating all non-negative values \( k_l \) and selection of those combinations that satisfy Eq. (1). This is achieved through the following algebraic structure:
\[ \nu_r(n) = \sum_{k_1 \ldots k_r} \delta(a_1 k_1 + \ldots + a_r k_r - n). \]  
The selection of the suitable combinations is carried out with the use of the Kronecker delta
\[ \delta(m - n) = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases} \]

We represent the Kronecker delta in the form of a contour integral:
\[ \delta(m - n) = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} z^m, \]
where the integration is counter clockwise in a neighbourhood of \( z = 0 \). With the use of this representation, Eq. (3) can be written in the form
\[ \nu_r(n) = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} \sum_{k_1 \ldots k_r} z^{a_1 k_1 + \ldots + a_r k_r}. \]  

For \( |z| < 1 \) the series converges, and the order of summation and integration can be changed. The summations over \( k_l \) are independent. There are \( r \) of the summations, and each is a geometric progression. The integral takes the form
\[ \nu_r(n) = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} \varphi(z), \]  
for the function \( \varphi(z) \) defined as
...
where

$$\varphi(z) = \prod_{l=1}^{r} \frac{1}{1 - z^{a_l}}.$$  \hfill (6)

The function $\varphi(z)$ is analogous to Euler’s generating function in the number theory and the probability generating function in the theory of probability. Expression (5) gives the derivative of $\varphi(z)$:

$$\nu_r(n) = \frac{1}{n!} \frac{d^n}{dz^n} \varphi(0).$$

Inside the unit circle the integrand $\varphi(z)/z^{n+1}$ is the analytic function with the pole at $z = 0$. To get a recursion, one can proceed by any of the following ways:

Firstly, one can integrate by parts. Secondly, one can exploit the analyticity of $\varphi(z)/z^{n+1}$. The radius of the path of integration in the neighbourhood $z = 0$ does not affect the result. Introducing the radius of the circle explicitly and differentiating it, we obtain a recursive formula that coincides with the formula obtained by using the integration by parts. Such techniques were used earlier, e.g. in Refs. [11, 12], to get recursive formulae for the normalization of particle-number-projected BCS wave function and for the probability distribution of the number of electron-positron pairs created in an external electric field. Thirdly, the integration over $dz/z$ is in fact the invariant integration in $U(1)$ group. By properties of the group integral, phase transformation $z \rightarrow ze^{i\chi}$ does not affect the value of integral. This yields an identity

$$\nu_r(n) = \frac{1}{2\pi i} \oint \frac{dz}{nz^n} \frac{d\ln(\varphi(z))}{dz} \varphi(z),$$  \hfill (7)

which is the special case of Ward identity.

The terms $1/(1 - z^{a_k})$ originating from the logarithmic derivative of $\varphi(z)$ can be expanded in a power series over $z^{a_k}$. The series is truncated because the singularity of the integrand is a finite-order pole at $z = 0$. The expansion terms of very high order remove such a singularity, so that the corresponding contour integrals vanish. The sum over $i$ is therefore within the limits $1 \leq i \leq [n/a_l]$. Comparing the different terms with Eq. (5), we notice that each term of the expansion represents $\nu_r(m)$ for some $m < n$. We thus arrive at Eq. (2).

The initial condition for the recursion $\nu_r(0) = 1$ is obvious. There are no solutions of the equation for negative $n$, so $\nu_r(n) = 0$ for $n < 0$.

Theorem 1 has

**Corollary** The recursive formula (2) can be written in the form

$$\nu_r(n) = \frac{1}{n} \sum_{m=1}^{n} \rho(m) \nu_r(n - m),$$  \hfill (8)
where

$$\rho(m) = \sum_{l=1}^{r} a_l \sum_{i \geq 1} \delta(m - ia_l)$$  \hspace{1cm} (9)$$

is the sum of coefficients $a_l$ that are divisors of $m$.

**Remark** Theorem 1 generalizes Theorem 15.1 of Ref. [2].

**Remark** A partition of $n$ is a nonincreasing sequence of positive integers whose sum equals $n$. For $a_l = l$, the number of partitions of $n$ is given by the corresponding coefficient in the expansion of the generating function (6) in a power series in the neighbourhood of $z = 0$ (Euler’s theorem). This number is equal to the number of distinct non-negative solutions of the Diophantine equation (2) with the coefficients $a_l = l$. The one-to-one correspondence between the partitions of $n$ and the solutions of the Diophantine equation (2) is achieved by interpreting the variable $k_l$ as the number of times of occurrence of the number $l$ in the partition of $n$.

**Remark** In the asymptotic regime $n \to \infty$, the summation over the index $i$ in Eq. (2) can be replaced by an integral. The derivative of Eq. (2) in $n$ leads then to an equation that has the solution

$$\nu_r(n) \sim C_r n^{r-1}.$$ 

The coefficient $C_r$ can be found from

$$\nu_r(n) = \sum_{i=0}^{[n/a_r]} \nu_{r-1}(n - ia_r).$$

In the continuum limit, the sum is replaced by an integral, which gives $C^{-1}_r = (r - 1)a_r C^{-1}_{r-1}$. By lowering further the index $r$, we obtain, for arbitrary coefficients $a_l$,

$$C^{-1}_r = (r - 1)! \prod_{l=1}^{r} a_l C^{-1}_0.$$ 

In the case where $a_l$ are coprime, $C_0 = 1$. [13]

If the coefficients contain only one common divisor, $d$, and $a_l/d$ are coprime, the problem reduces to the case of the coprime coefficients. In such a case, $C_0 = d$, when the $n = 0 \pmod{d}$ and $C_0 = 0$, when $n \neq 0 \pmod{d}$. After averaging $n$ over an interval $\Delta n > d$, we get $C_1 = 1$. The equality $C_0 = 1$ also holds when only a part of the coefficients has a single common divisor. To see this, we use the equation

$$\nu_{p+q}(n) = \sum_{s=0}^{[n/d]} \nu_p(n - ds) \nu_q(s).$$

A similar recursion can be found in Ref. [9]. The first $p$ coefficients are coprime, while the last $q$ coefficients have a single common divisor $d$. $\nu_q(s)$ counts the number of solutions of Eq. (1) with the integer coprime coefficients $a_l/d (l = p+1, \ldots, p+q)$. 


Example Consider the problem of finding the number of distinct terms in the expansion of the determinant in the sum of products of traces of powers of the matrix. The determinant is represented as follows [14]

$$\det \|A\| = \sum_{k_1 \ldots k_n} \prod_{i=1}^{n} \frac{(-1)^{k_i+1}}{l_i^k_i} \text{tr}(A_i^{k_i})$$

where the admissible sets of non-negative $k_i$ over which we take the summation are determined by solutions of Eq. (1) with $a_l = l$ and $r = n$. The number of the various terms is given by Eq. (2). For matrices with low dimensionality, we obtain $\nu_n(n) = 2, 3, 5, 7, 11, 15, 22, \ldots$ for $n = 2, 3, 4, 5, 6, 7, 8, \ldots$, respectively. As previously noted, the number of solutions of Eq. (2) with the coefficients $a_l = l$ coincides with the number of partitions of $n$. The asymptotic behaviour of $\nu_n(n)$ for $n \to \infty$ has the form [2]

$$\nu_n(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi \sqrt{2n/3}).$$

The number of terms on the right side of Eq. (10) is growing sub-exponentially. In terms of computing, the most economical method to calculate determinants is the Gauss elimination method, which requires a polynomially large number of operations. The decomposition (10) is of interest when there is a symmetry and a need to preserve it at each step of the calculation.

Example A similar combinatorial problem arises in calculating the derivative of composite functions. This problem leads to the Fa`a di Bruno’s formula (see, e.g., [15])

$$\frac{d^r}{dz^r} f(g(z)) = \sum_{k_1 \ldots k_r} f^{(k_1+k_2+\ldots+k_r)}(g(z)) \prod_{l=1}^{r} \frac{1}{l_i^{k_i}k_i!} (g^{(l)}(z))^{k_i},$$

where $f^{(m)}$ and $g^{(m)}$ are the $m$-order derivatives; the summation is over all sets of the non-negative $k_1, \ldots, k_r$ that satisfy Eq. (1) with $a_l = l$. The number of terms in the right side of Eq. (11) and the asymptotic behavior are those of the trace decomposition (10).

Example As another application, one can mention the Bell polynomials [16] in terms of which the right sides of Eqs. (10) and (11) can be expressed. The number of terms in the $n$-th complete Bell polynomial is equal to the number of solutions to Eq. (1) with $a_l = l$ and $r = n$.

Example For the case of $a_l = 1$ one may find an explicit expression for $\nu_r(n)$ from Eq. (4). By moving the contour to infinity, we obtain

$$\nu_r(n) = \frac{(n+r-1)!}{n!(r-1)!}.$$
For \( a_l = 1 \) this equation solves the recursion (2). It can be noted that \( \nu_r (n) \) is equal to the number of independent components of a rank-\( n \) symmetric tensor in the space of dimension \( r \). The solutions \( a_l = 1 \) differ from the solutions \( a_l = l \) combinatorially in the sense that all the partitions of \( a_l = 1 \) are considered to be different. For instance, \( 1 + 1 + 3 = 5 \) and \( 1 + 3 + 1 = 5 \) are counted as distinct partitions of 5, whereas in the case of \( a_l = l \) these partitions are counted as the one with \( k_1 = 2, k_2 = 0, k_3 = 1 \).

**Example** We consider the random walk of a particle on the one-dimensional lattice. Suppose that the probability distribution in one step is described by the Poisson distribution

\[
p_1(n) = \frac{\alpha^n}{n!} \exp(-\alpha) .
\]

(13)

The parameter \( \alpha \) is the average displacement in one step. Condition \( a_l > 0 \) means that the particle is moving forward. The probability generating function is of the form

\[
\varphi(z) = \exp \left( \sum_{l=1}^{r} (z^{a_l} - 1)\alpha \right) .
\]

(14)

The probability of displacement in \( r \) steps by distance \( n \) is given by the right side of Eq. (5). Applying the Ward identity, we find a recursive formula

\[
p_r(n) = \alpha \sum_{l=1}^{r} a_l p_r(n - a_l) .
\]

(15)

The initial conditions are \( p_r(n) = 0 \) for \( n < 0 \) and \( p_r(0) = (p_1(0))^r = \exp(-\alpha r) \).

### 3. QUADRATIC DIOPHANTINE EQUATION

We consider the quadratic Diophantine equation

\[
a_1 k_1^2 + \ldots + a_r k_r^2 = n
\]

(16)

for positive integer coefficients \( a_l \in \mathbb{N} \) and \( k_l \in \mathbb{Z} (l = 1, \ldots, r) \). The main result of this section can be summarized in

**Theorem 2** The number of solutions of the Diophantine equation (16) with positive coefficients \( a_l \) can be obtained with the help of the recursive formula

\[
\nu_r(n) = \frac{1}{2n} \sum_{l=1}^{r} a_l \sum_{pq} \left( -1 + (-1)^{p-1} + 2(-1)^{q-1} + 2(-1)^{p+q} \right) \nu_r(n - a_l pq) .
\]

(17)

The summation is over the indices \( l \) from one to \( r \) and \( p \) and \( q \) from one to \( pq \leq \lfloor n/a_l \rfloor \). The initial condition is \( \nu_r(0) = 1 \), and \( \nu_r(n) \) is taken to be 0 if \( n < 0 \).
Proof The number of solutions of (16) is written as a contour integral
\[
\nu_r(n) = \frac{1}{2\pi i} \oint \frac{dz}{zn+1} \sum_{k_1 \ldots k_r} z^{a_1 k_1^2 + \ldots + a_r k_r^2}.
\] (18)

This representation is valid for \(a_l > 0\), because the series with a non-positive \(a_l\) diverges, when the series with positive \(a_l\) converge, and vice versa.

The peculiarity of the recursion is the generating function
\[
\varphi(z) = \prod_{l=1}^{r} \vartheta(1, z^{a_l}),
\]
expressed in terms of the product of Jacobi theta functions. Using the method of Sect. 2, we obtain
\[
\nu_r(n) = \frac{1}{2\pi i} \oint \frac{dz}{nz^n} \sum_{l=1}^{r} a_l z^{a_l-1} \frac{\vartheta'(1, z^{a_l})}{\vartheta(1, z^{a_l})} \varphi(z).
\] (19)

The triple product identity
\[
\vartheta(w, u) = \prod_{m=1}^{\infty} \left( 1 - u^{2m} \right) \left( 1 + w^2 u^{2m-1} \right) \left( 1 + w^{-2} u^{2m-1} \right)
\]
allows to get the series expansion of the logarithmic derivative in the neighborhood of \(u = 0\):
\[
\frac{\vartheta'(1, u)}{\vartheta(1, u)} = \sum_{m=1}^{+\infty} \sum_{s=0}^{+\infty} u^{2m(s+1)-1} \left( (-2m) + 2(2m - 1)(-1)^s u^{-(s+1)} \right)
\]
Substituting this expression for \(u = z^{a_l}\) in Eq. (19), we arrive at Eq. (17).

Example Using Eq. (17), we obtain for \(a_l = 1\) \(\nu_2(n) = 1, 4, 4, 0, 4, 8, 0, 4, 8\) and \(\nu_3(n) = 1, 6, 12, 8, 6, 24, 24, 0, 12, 30, 24\) for \(n = 0, 1, \ldots, 10\), which is in the agreement with the direct expansion of the generating functions.

4. DIOPHANTINE EQUATION OF GENERAL ADDITIVE FORM

Now consider the general case:
\[
g_1(k_1) + \ldots + g_r(k_r) = n,
\] (20)
for \(k_l \in \mathbb{N}_0\). The number of solutions, \(\nu_r(n)\), is finite provided \(g_l(k) \in \mathbb{N}_0\). We also consider the case of increasing functions: \(\forall l : k < m \iff g_l(k) < g_l(m)\). Without loss of generality, one can assume \(g_l(0) = 0\).

Theorem 3 Under the specified conditions the number of solutions of Eq. (20) can
be calculated from the recursion

$$\nu_r(n) = \frac{1}{n} \sum_{l=1}^{r} \sum_{m=1}^{n} \frac{1}{(m-1)!} K_m(c_{l1}, \ldots, c_{lm}) \nu_r(n-m), \quad (21)$$

where

$$K_n(c_{l1}, \ldots, c_{ln}) = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! B_{n,k}(1!c_{l1}, \ldots, (n-k+1)!c_{ln-k+1}),$$

$B_{n,k}(x_1, \ldots, x_{n-k+1})$ are the partial Bell polynomials, and $c_{lk}$ are the expansion coefficients of the generating functions for the sequences of individual terms in Eq. (20):

$$\phi_l(z) \equiv \sum_{k=0}^{\infty} z^{g_l(k)} = 1 + \sum_{k=1}^{\infty} c_{lk} z^k. \quad (22)$$

The initial conditions are as follows: $\nu_r(n) = 0$ for $n < 0$ and $\nu_r(0) = 1$.

**Proof** The number of solutions is calculated by the same method as in the previous two sections. The expansion coefficients of the generating functions (22) are given by:

$$c_{lk} = \begin{cases} 1, & \exists m > 0 : k = g_l(m), \\ 0, & \forall m > 0 : k \neq g_l(m). \end{cases}$$

The logarithmic derivative of the full generating function

$$\varphi(z) = \prod_{l=1}^{r} \varphi_l(z) \quad (23)$$

enters the integral form of $\nu_r(n)$. For the logarithm of $\varphi_l(z)$, we have the following representation

$$\ln \left( \varphi_l(z)e^C \right) = \int_{0}^{\infty} \frac{d\xi}{\xi} \left( -\exp(-\xi \varphi_l(z)) + \frac{1}{1+\xi} \right), \quad (24)$$

where $C = 0.577 \ldots$ is the Euler constant. The exponent is expanded in a neighbourhood of $z = 0$ in terms of the partial Bell polynomials [16]

$$\exp(-\xi \varphi_l(z)) = \exp(-\xi) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} (-\xi)^k B_{n,k}(1!c_{l1}, \ldots, (n-k+1)!c_{ln-k+1}) z^n$$

Taking the derivative on both sides of Eq. (24), changing the order of summation and integration, and integrating over $\xi$, we obtain

$$\frac{\varphi_l'(z)}{\varphi_l(z)} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} K_n(c_{l1}, \ldots, c_{ln}) z^{n-1}.$$
Remark The above scheme is limited by the requirement of positive $g_l(k_l)$. This constraint can be eliminated provided the additivity holds. Let us consider the equation

$$g_1(k_1) + \ldots + g_r(k_r) = g_{r+1}(k_{r+1}) + \ldots + g_{r+s}(k_{r+s}), \quad (25)$$

with non-negative functions $g_l(k_l)$ ($l = 1, \ldots, r + s$). If some functions in equation (20) are negative, they can be placed in the right side, in which case we arrive at Eq. (25). The number of solutions of Eq. (25) is the same as the number of solutions in the system of two equations, Eq. (20) and

$$g_{r+1}(k_{r+1}) + \ldots + g_{r+s}(k_{r+s}) = n, \quad (26)$$

where the right side parameter is not fixed. We have compare solutions of equations (20) and (26) for all values of the parameter $n$. The number of solutions of these equations equals $\nu_r(n)$ and $\nu_s(n)$, respectively. The number of solutions of the system is the product $\nu_r(n)\nu_s(n)$ summed over all $n$, and it can diverge.

Example We illustrate the method by finding few lowest solutions of the equation

$$k_1^3 + k_2^3 = k_3^2. \quad (27)$$

The right side is considered as a parameter $n$. If the number of solutions for a square $n$ is different from zero, we get the proof on the existence of the solution. Bell polynomials are programmed as standard functions with Maple 15. Using the recursion (21), we obtain $\nu_2(1) = \nu_2(8) = \nu_2(9) = 2, \; \nu_2(2) = \nu_2(16) = 1$ and $\nu_2(n) = 0$ in other cases, for $n = 1, \ldots, 50$. The first two solutions of Eq. (27) correspond to $k_1 = 1, k_2 = 2, n = 3^2$ and $k_1 = 2, k_2 = 2, n = 4^2$.

A slight modification of the arguments leads to

**Theorem 4** Let $d_{lk}$ be expansion coefficients of the logarithm of the generating functions (22). The expansion coefficients of the logarithm of the full generating function (23) are given then by

$$d_k = \sum_{l=1}^{r} d_{lk}, \quad (28)$$

while the number of solutions of Eq. (20) is given by the $n$-th complete Bell polynomial:

$$\nu_r(n) = \frac{1}{n!} B_n(1!d_1, \ldots, n!d_n). \quad (29)$$

Proof The expansion of generating function in the neighbourhood of $z = 0$ determines the expansion of its logarithm:

$$1 + \sum_{k=1}^{\infty} c_{lk}z^k = \exp(\sum_{k=1}^{\infty} d_{lk}z^k). \quad (30)$$
We write the contour integral of both sides of this equation. The Ward identity leads to the relationship
\[ c_{ln} = d_{ln} + \sum_{k=1}^{n-1} \frac{k}{n} d_{lk} c_{ln-k}, \]  
(31)
which is commonly used in probability theory for calculation of cumulants. This recursion is bilateral: it allows to find expansion coefficients of the left side of Eq. (30) in terms of expansion coefficients of the right side, and vice versa (allowing thereby to compute \( \nu_r(n) \) with at most \( O(rn^2) \) operations). Expanding the exponential representation of \( \varphi(z) \) over the complete Bell polynomials, we obtain the expression (29).

**Corollary** Substituting the logarithmic derivative of (23) into Eq. (7), we obtain
\[ \nu_r(n) = \frac{1}{n} \sum_{k=1}^{n} kd_k \nu_r(n-k). \]  
(32)

**Remark** Equation (29) solves the recursions (2), (17), (21), and (32). These recursions appear as identities for the complete Bell polynomials.

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