The goal of this paper is to analyze, using homogenization techniques, the effective thermal transfer in a periodic composite material formed by two constituents, separated by an imperfect interface. The imperfect contact between the constituents generates a contact resistance and, depending on the magnitude of this resistance, a threshold phenomenon arises.

Key words: homogenization, composite materials, imperfect interfaces.

PACS: 44.05.+e, 44.10.+i, 44.30.+v, 44.35.+c, 81.05.Rm.

1. INTRODUCTION

In the last decades, the problem of thermal transfer in heterogeneous media has been a subject of huge interest for a broad category of researchers: engineers, mathematicians, physicists (see [2] and [12]). Also, addressing contact problems for multiphase composites is important, since it is known that the macroscopic properties of a composite can be affected by the imperfect bonding between its constitutive components (for a review of the literature on imperfect interfaces in heterogeneous media, we refer to [13] and [15]).

The main goal of this paper is to describe the macroscopic behavior of a system of coupled partial differential equations arising in the modeling of thermal transport in a two-component composite. We deal, at the microscale, with a periodic structure formed by two connected media with different thermal properties, separated by an imperfect interface. We assume that we have nonlinear sources acting in each media and that at the interface between the two constituents the flux is continuous, but the temperature field has a jump. We are interested in describing the asymptotic behavior, as the small parameter which characterizes the sizes of the two constituents tends to zero, of the temperature field in the periodic composite. The imperfect contact between the constituents generates a contact resistance and, depending on the magnitude of this resistance, a threshold phenomenon arises. So, depending on the rate exchange between the two phases, three important cases are considered and three
different types of limit problems are obtained from the same type of micromodel (see Section 3).

For simplicity, we deal here only with the stationary case, but we mention that the dynamic one can be treated in a similar manner (see [8] and [14]). Our setting can be also relevant for studying electrical conduction in biological tissues (see [1], [17] and [19]).

Our approach is based on the periodic unfolding method, recently introduced by D. Cioranescu, A. Damlamian, G. Griso, P. Donato and R. Zaki (see [6] and [7]). An advantage offered by our approach is that we can avoid the use of extension operators and, therefore, we can deal rigorously with media with less regularity than those usually considered in the literature.

Similar problems have been addressed, using different techniques, formal or not, in [2], [3], [13] and [11]. Our approach, as already mentioned, is based on a different method, the periodic unfolding method, which allows us to deal with more general media. The results presented in this paper also constitute a generalization of those obtained in [11, 14, 18, 19]. Corrector results and results for the case of nonsymmetric matrices will be presented in a future paper.

For heat conduction problems in a periodic material with a different geometry, we refer to [9] and [16] and the references therein.

The plan of the paper is as follows: in the second section, we formulate the microscopic problem. In the third section, we give our main results, while the last section is devoted to the proof of the convergence results. The paper ends with a few conclusions and some references.

2. PROBLEM SETTING

Let \( \Omega \) be an open bounded material body in \( \mathbb{R}^n \) (\( n \geq 3 \)), with a Lipschitz-continuous boundary \( \partial \Omega \). We assume that \( \Omega \) is formed by two constituents, \( \Omega_1^\varepsilon \) and \( \Omega_2^\varepsilon \), representing two materials with different thermal characteristics, separated by an imperfect interface \( \Gamma^\varepsilon \). We also assume that both phases \( \Omega_1^\varepsilon \) and \( \Omega_2^\varepsilon = \Omega \setminus \Omega_1^\varepsilon \) are connected, but only \( \Omega_1^\varepsilon \) reaches the external fixed boundary \( \partial \Omega \). Here, \( \varepsilon \) represents a small parameter related to the characteristic size of the two constituents. Let \( Y_1 \) be a Lipschitz open connected subset of the unit cell \( Y = (0,1)^n \) and \( Y_2 = Y \setminus Y_1 \). We assume that \( Y_2 \) has a locally Lipschitz boundary and the intersections of the boundary of \( Y_2 \) with the boundary of \( Y \) are identically reproduced on opposite faces of the cell. Also, we suppose that, repeating \( Y \) by periodicity, the union of all the sets \( \overline{Y_1} \) is connected and has a locally \( C^2 \) boundary (see [11]).

Let

\[
Z_\varepsilon = \{ k \in \mathbb{Z}^n \mid \varepsilon k + \varepsilon Y \subseteq \Omega \},
\]

\[
K_\varepsilon = \{ k \in Z_\varepsilon \mid \varepsilon k \pm \varepsilon e_i + \varepsilon Y \subseteq \Omega, \forall i = 1, n \}.
\]
where $e_i$ are the elements of the canonical basis of $\mathbb{R}^n$.

We define

$$\Omega_2^\varepsilon = \text{int}(\bigcup_{k \in K_\varepsilon} (\varepsilon k + \varepsilon Y_2)), \quad \Omega_1^\varepsilon = \Omega \setminus \Omega_2^\varepsilon$$

and we set $\theta = |Y \setminus Y_2|$.

For $\alpha_1, \beta_1 \in \mathbb{R}$, with $0 < \alpha_1 < \beta_1$, let $\mathcal{M}(\alpha_1, \beta_1, Y)$ be the set of all the square matrices $A \in (L^\infty(Y))^{n \times n}$ such that, for any $\xi \in \mathbb{R}^n$, $(A(y)\xi, \xi) \geq \alpha_1 |\xi|^2$, $|A(y)\xi| \leq \beta_1 |\xi|$, almost everywhere in $Y$. Let $A^\varepsilon(x) = A(x/\varepsilon)$ defined on $\Omega$, where $A \in \mathcal{M}(\alpha_1, \beta_1, Y)$ is a symmetric smooth $Y$-periodic matrix. We shall denote the matrix $A$ by $A_1$ in $Y_1$ and by $A_2$, respectively, in $Y_2$.

Our goal is to describe the effective behavior of the solution $(u^\varepsilon, v^\varepsilon)$ of the following coupled system of equations:

$$\begin{cases}
-\text{div} (A_1^\varepsilon \nabla u^\varepsilon) + \alpha(u^\varepsilon) = f & \text{in } \Omega_1^\varepsilon, \\
-\text{div} (A_2^\varepsilon \nabla v^\varepsilon) + \beta(v^\varepsilon) = f & \text{in } \Omega_2^\varepsilon, \\
A_1^\varepsilon \nabla u^\varepsilon \cdot \nu = A_2^\varepsilon \nabla v^\varepsilon \cdot \nu & \text{on } \Gamma^\varepsilon, \\
A_1^\varepsilon \nabla u^\varepsilon \cdot \nu = \varepsilon \gamma h(u^\varepsilon, v^\varepsilon) & \text{on } \Gamma^\varepsilon, \\
u^\varepsilon = 0 & \text{on } \partial\Omega.
\end{cases} \tag{1}$$

Here, $\nu$ is the unit outward normal to $\Omega_1^\varepsilon$ and $f \in L^2(\Omega)$.

Thus, we consider that the flux is continuous across the boundary $\Gamma^\varepsilon$, but, since the interface between the two phases is not perfect, the continuity of temperatures is replaced by a Biot boundary condition.

We assume that the functions $\alpha = \alpha(r)$ and $\beta = \beta(r)$ are continuous, monotonously non-decreasing with respect to $r$ and such that $\alpha(0) = 0$ and $\beta(0) = 0$. Moreover, we suppose that there exist $C \geq 0$ and an exponent $q$, with $0 \leq q < n/(n-2)$, such that

$$|\alpha(r)| \leq C(1 + |r|^q) \tag{2}$$

and

$$|\beta(r)| \leq C(1 + |r|^q). \tag{3}$$

We also assume that

$$h(u^\varepsilon, v^\varepsilon) = h_0^\varepsilon(x)(v^\varepsilon - u^\varepsilon) \tag{4}$$

where $h_0^\varepsilon(x) = h_0\left(x/\varepsilon\right)$ and $h_0(y)$ is a $Y$-periodic, smooth real function with $h_0(y) \geq \delta > 0$. Moreover, we consider that

$$H = \int_{\Gamma} h_0(y) \, d\sigma \neq 0.$$
Let us notice that, following [14], we can treat in a similar manner the case in which the function \( h \) is of the following relevant form:

\[
h(r,s) = \bar{h}(r,s)(s-r),
\]

with \( 0 < h_{\text{min}} \leq \bar{h}(r,s) \leq h_{\text{max}} < \infty \). Also, we can deal with the more general case in which the nonlinear functions \( \alpha \) and \( \beta \) are multi-valued maximal monotone graphs, as in [8].

For results concerning the well posedness of problem (1), we refer to [1], [14] and [17–19].

Since it is impossible to solve this system, the microstructure must be homogenized in order to obtain a new model that describes its macroscopic properties. Using the periodic unfolding method introduced by D. Cioranescu, A. Damlamian, G. Griso, P. Donato and R. Zaki (see [6] and [7]), we can describe the asymptotic behavior of the solution of system (1). This behavior depends on the values of the parameter \( \gamma \), i.e., on the contact resistance between the two constituents. There are three interesting cases to be considered: a) \( \gamma = 1 \); b) \( \gamma = 0 \); c) \( \gamma = -1 \).

In the most interesting case, \( \gamma = 1 \), we obtain at the limit a new nonlinear system (see (5)). At a macroscopic scale, the composite medium can be represented by a continuous model, which conceives it as the superimposition of two interpenetrating continuous media, coexisting at every point of the domain. For the other two cases, we obtain at the limit only one equation (see (6) and (7)).

3. THE MAIN RESULTS

In this section, we shall describe the effective behavior of the solutions of the microscopic model (1) for the above mentioned three cases.

a) Let us consider first the case \( \gamma = 1 \).

**Theorem 1.** For \( \gamma = 1 \), the solution \((u^\varepsilon, v^\varepsilon)\) of system (1) converges, as \( \varepsilon \to 0 \), to the unique solution \((u, v)\), with \( u, v \in H^1_0(\Omega) \), of the following macroscopic problem:

\[
\begin{align*}
-\div (\bar{\mathbf{A}}^1 \nabla u) + \theta \alpha(u) - H(v-u) &= \theta f \quad \text{in } \Omega, \\
-\div (\bar{\mathbf{A}}^2 \nabla v) + (1-\theta) \beta(v) + H(v-u) &= (1-\theta)f \quad \text{in } \Omega.
\end{align*}
\]

In (5), \( \bar{\mathbf{A}}^1 \) and \( \bar{\mathbf{A}}^2 \) are the homogenized matrices, defined by:

\[
\bar{\mathbf{A}}^1_{ij} = \int_{Y_1} \left( a_{ij} + a_{ik} \frac{\partial \chi_{1j}}{\partial y_k} \right) dy,
\]

\[
\bar{\mathbf{A}}^2_{ij} = \int_{Y_2} \left( a_{ij} + a_{ik} \frac{\partial \chi_{2j}}{\partial y_k} \right) dy
\]
and $\chi_{1k} \in H^1_{\text{per}}(Y_1)/\mathbb{R}$, $\chi_{2k} \in H^1_{\text{per}}(Y_2)/\mathbb{R}$, $k = 1, \ldots, n$, are the weak solutions of the cell problems

\[
\begin{align*}
-\nabla_y \cdot (A_1(y) \nabla_y \chi_{1k}) &= \nabla_y A_1(y) e_k, \quad y \in Y_1, \\
(A_1(y) \nabla_y \chi_{1k}) \cdot \nu &= -A_1(y) e_k \cdot \nu, \quad y \in \Gamma,
\end{align*}
\]

\[
\begin{align*}
-\nabla_y \cdot (A_2(y) \nabla_y \chi_{2k}) &= \nabla_y A_2(y) e_k, \quad y \in Y_2, \\
(A_2(y) \nabla_y \chi_{2k}) \cdot \nu &= -A_2(y) e_k \cdot \nu, \quad y \in \Gamma.
\end{align*}
\]

So, at a macroscopic scale, the composite medium, despite of its discrete structure, can be represented by a continuous model, which is similar to the so-called bi-domain model, arising in the context of diffusion in partially fissured media (see [4] and [11]) or in the case of electrical activity of the heart (see [1] and [17]).

b) For $\gamma = 0$, i.e. for high contact resistance, we get, at the macroscale, only one temperature field. So, $u = v = u^0 \in H^1_0(\Omega)$ and $u^0$ satisfies:

\[
-\text{div} (A^0 \nabla u^0) + \theta \alpha(u^0) + (1 - \theta) \beta(u^0) = f \text{ in } \Omega. \tag{6}
\]

Here, the effective matrix $A^0$ is given by:

\[
A^0_{ij} = \int_{Y_1} (a_{ij} + a_{ik} \frac{\partial \chi_{1k}}{\partial y_k}) \, dy + \int_{Y_2} (a_{ij} + a_{ik} \frac{\partial \chi_{2k}}{\partial y_k}) \, dy,
\]

in terms of the functions $\chi_{1k} \in H^1_{\text{per}}(Y_1)/\mathbb{R}$, $\chi_{2k} \in H^1_{\text{per}}(Y_2)/\mathbb{R}$, $k = 1, \ldots, n$, weak solutions of the cell problems

\[
\begin{align*}
-\nabla_y \cdot (A_1(y) \nabla_y \chi_{1k}) &= \nabla_y A_1(y) e_k, \quad y \in Y_1, \\
(A_1(y) \nabla_y \chi_{1k}) \cdot \nu &= -A_1(y) e_k \cdot \nu, \quad y \in \Gamma,
\end{align*}
\]

\[
\begin{align*}
-\nabla_y \cdot (A_2(y) \nabla_y \chi_{2k}) &= \nabla_y A_2(y) e_k, \quad y \in Y_2, \\
(A_2(y) \nabla_y \chi_{2k}) \cdot \nu &= -A_2(y) e_k \cdot \nu, \quad y \in \Gamma.
\end{align*}
\]

Let us notice that in this case, the insulation provided by the interface is sufficient to modify the limiting diffusion matrix, but it is not strong enough to force the existence of two different limit phases.

c) For the case $\gamma = -1$, i.e. for weak contact resistance, we also get, at the limit, $u = v = u_0 \in H^1_0(\Omega)$ and, in this case, the effective temperature field $u_0$ satisfies:

\[
-\text{div} (A_0 \nabla u_0) + \theta \alpha(u_0) + (1 - \theta) \beta(u_0) = f \text{ in } \Omega. \tag{7}
\]
The macroscopic coefficients are given by:

\[ A_{0,ij} = \int_{Y_1} \left( a_{ij} + a_{ik} \frac{\partial w_{1j}}{\partial y_k} \right) \, dy + \int_{Y_2} \left( a_{ij} + a_{ik} \frac{\partial w_{2j}}{\partial y_k} \right) \, dy, \]

where \( w_{1k} \in H^1_{per}(Y_1)/\mathbb{R}, \, w_{2k} \in H^1_{per}(Y_2)/\mathbb{R}, \, k = 1, \ldots, n, \) are the weak solutions of the cell problems

\[
\begin{align*}
-\nabla_y \cdot (A_1(y) \nabla_y w_{1k}) &= \nabla_y A_1(y) e_k, \quad y \in Y_1, \\
-\nabla_y \cdot (A_2(y) \nabla_y w_{2k}) &= \nabla_y A_2(y) e_k, \quad y \in Y_2, \\
(A_1(y) \nabla_y w_{1k}) \cdot \nu &= (A_2(y) \nabla_y w_{2k}) \cdot \nu, \quad y \in \Gamma, \\
A_1(y) \nabla_y w_{1k} \cdot \nu + h_0(y)(w_{1k} - w_{2k}) &= -A_1(y) e_k \cdot \nu, \quad y \in \Gamma.
\end{align*}
\]

We remark that in this case, the effective coefficients depend on \( h_0. \)

## 4. PROOF OF THE MAIN RESULTS

We shall sketch now the proof for the main case, i.e. \( \gamma = 1. \) Let

\[ H^1_{\partial \Omega}(\Omega^\varepsilon_1) = \{ v \in H^1(\Omega^\varepsilon_1) \mid v = 0 \text{ on } \partial \Omega \cap \partial \Omega^\varepsilon_1 \}, \]

endowed with the norm \( \| v \|_{H^1_{\partial \Omega}(\Omega^\varepsilon_1)} = \| \nabla v \|_{L^2(\Omega^\varepsilon_1)}. \)

Let \( H^\varepsilon = H^1_{\partial \Omega}(\Omega^\varepsilon_1) \times H^1(\Omega^\varepsilon_2), \) endowed with the scalar product

\[
(u, \varphi)_\varepsilon = \int_{\Omega^\varepsilon_1} \nabla u_1 \cdot \nabla \varphi_1 \, dx + \int_{\Omega^\varepsilon_2} \nabla u_2 \cdot \nabla \varphi_2 \, dx + \varepsilon \int_{\Gamma^\varepsilon} (u_1 - u_2)(\varphi_1 - \varphi_2) \, d\sigma, \quad \forall u = (u_1, u_2), \varphi = (\varphi_1, \varphi_2) \in H^\varepsilon. \tag{8}
\]

The norm associated to the scalar product (8) is equivalent to the standard norm in \( H^1_{\partial \Omega}(\Omega^\varepsilon_1) \times H^1(\Omega^\varepsilon_2), \) with constants independent of \( \varepsilon \) (see [9] and [11]).

Let us give now the variational formulation of problem (1).

Find \( (u^\varepsilon, v^\varepsilon) \in H^\varepsilon \) such that

\[
\int_{\Omega^\varepsilon_1} A_1^\varepsilon \nabla u^\varepsilon \cdot \nabla \varphi_1 \, dx + \int_{\Omega^\varepsilon_2} A_2^\varepsilon \nabla v^\varepsilon \cdot \nabla \varphi_2 \, dx + \int_{\Omega^\varepsilon_1} \alpha(u^\varepsilon) \varphi_1 \, dx + \int_{\Omega^\varepsilon_2} \beta(v^\varepsilon) \varphi_2 \, dx \\
+ \varepsilon \int_{\Gamma^\varepsilon} h(u^\varepsilon, v^\varepsilon)(\varphi_1 - \varphi_2) \, d\sigma = \int_{\Omega^\varepsilon_1} f \varphi_1 \, dx + \int_{\Omega^\varepsilon_2} f \varphi_2 \, dx, \tag{9}
\]

for any \( \varphi = (\varphi_1, \varphi_2) \in H^\varepsilon. \)

The problem (9) has a unique weak solution \( (u^\varepsilon, v^\varepsilon) \in H^\varepsilon \) (see, for instance, [9], [11], [14] and [18]).
using the hypotheses we imposed on the data, we can obtain suitable energy estimates, independent of \( \varepsilon \), for our solution (see [8], [18] and [19]). We get:

\[
\int_{\Omega_1^\varepsilon} A_1^\varepsilon \nabla u^\varepsilon \cdot \nabla \eta_1 \, dx + \int_{\Omega_2^\varepsilon} A_2^\varepsilon \nabla v^\varepsilon \cdot \nabla \eta_2 \, dx + \varepsilon \int_{\Gamma^\varepsilon} h(u^\varepsilon, v^\varepsilon)(v^\varepsilon - u^\varepsilon) \, d\sigma \leq C,
\]

where \( C \) is independent of \( \varepsilon \). So, \((u^\varepsilon, v^\varepsilon)\) is bounded in \( H^2 \) and there exists a constant \( C > 0 \), independent of \( \varepsilon \), such that

\[
\|u^\varepsilon\|_{H^1(\Omega_1)} \leq C,
\]

\[
\|v^\varepsilon\|_{H^1(\Omega_1^\varepsilon)} \leq C
\]

and

\[
\|v^\varepsilon - u^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq C \varepsilon^{-\frac{1}{2}}.
\]

For obtaining the macroscopic problem (5), we shall make use of the unfolding operators \( T_1^\varepsilon \) and \( T_2^\varepsilon \) introduced in [6] and [9]. Since these operators map functions defined on the oscillating domains \( \Omega_1^\varepsilon \) and \( \Omega_2^\varepsilon \) into functions defined on the fixed domains \( \Omega \times Y_1 \) and \( \Omega \times Y_2 \), respectively, we can avoid the use of extension operators.

Using our a priori estimates and the properties of the above mentioned unfolding operators, we can prove that there exist \( u, v \in H_0^1(\Omega), \hat{u} \in L^2(\Omega; H^{1,\text{per}}_1(Y_1)), \hat{v} \in L^2(\Omega; H^{1,\text{per}}_2(Y_2)) \) such that, up to a subsequence, for \( \varepsilon \to 0 \), we have:

\[
T_1^\varepsilon(u^\varepsilon) \to u \text{ weakly in } L^2(\Omega, H^1(Y_1)),
\]

\[
T_1^\varepsilon(\nabla u^\varepsilon) \to \nabla u + \nabla_y \hat{u} \text{ weakly in } L^2(\Omega \times Y_1),
\]

\[
T_2^\varepsilon(v^\varepsilon) \to v \text{ weakly in } L^2(\Omega, H^1(Y_2)),
\]

\[
T_2^\varepsilon(\nabla v^\varepsilon) \to \nabla v + \nabla_y \hat{v} \text{ weakly in } L^2(\Omega \times Y_2).
\]

Let us take now \( \Phi_1, \Phi_2 \in C_0^\infty(\Omega) = D(\Omega) \) as test functions in (9). We have:

\[
\int_{\Omega_1^\varepsilon} A_1^\varepsilon \nabla u^\varepsilon \cdot \nabla \Phi_1 \, dx + \int_{\Omega_2^\varepsilon} A_2^\varepsilon \nabla v^\varepsilon \cdot \nabla \Phi_2 \, dx +
\]

\[
\int_{\Omega_1^\varepsilon} \alpha(u^\varepsilon)\Phi_1 \, dx + \int_{\Omega_2^\varepsilon} \beta(v^\varepsilon)\Phi_2 \, dx + \varepsilon \int_{\Gamma^\varepsilon} h(u^\varepsilon, v^\varepsilon)(\Phi_2 - \Phi_1) \, d\sigma =
\]

\[
\int_{\Omega_1^\varepsilon} f\Phi_1 \, dx + \int_{\Omega_2^\varepsilon} f\Phi_2 \, dx.
\]

We intend now to apply the corresponding unfolding operators in (10) and to pass to the limit, with \( \varepsilon \to 0 \). For passing to the limit in the terms involving the functions \( \alpha, \beta \) and \( h \), let us notice that, exactly like in [8] and [19], one can prove that, assuming (2)-(4) and using the properties of the unfolding operators \( T_1^\varepsilon \) and \( T_2^\varepsilon \) (see, for instance, [7]), one gets:

\[
\int_{\Omega_1^\varepsilon} \alpha(u^\varepsilon)\Phi_1 \, dx \to \int_{\Omega \times Y_1} \alpha(u)\Phi_1 \, dx \, dy,
\]
Thus, applying the unfolding operators and passing to the limit in (10), we get:
\[
\int_{\Omega \times Y_1} A_1(\nabla u + \nabla y \hat{\nu}) \cdot \nabla \Phi_1 \, dy + \int_{\Omega \times Y_2} A_2(\nabla v + \nabla y \tilde{\nu}) \cdot \nabla \Phi_2 \, dy + \\
\int_{\Omega \times Y_1} \alpha(u) \Phi_1 \, dx dy + \int_{\Omega \times Y_2} \beta(v) \Phi_2 \, dx dy + \\
H \int_{\Omega} (v - u)(\Phi_2 - \Phi_1) \, dx = \int_{\Omega \times Y_1} f \Phi_1 \, dx dy + \int_{\Omega \times Y_2} f \Phi_2 \, dx dy.
\]
(11)

Now, we take in (9) the test functions \( w_i^\varepsilon = \varepsilon \Phi_i(x) \varphi_i \left( \frac{x}{\varepsilon} \right) \), with \( i = 1, 2 \), where \( \Phi_i \in D(\Omega) \), \( \varphi_i \in H^1_{\text{per}}(Y_i) \). Obviously, \( T_i^\varepsilon( w_i^\varepsilon) \to 0 \), strongly in \( L^2(\Omega \times Y_i) \) and \( T_i^\varepsilon(\nabla w_i^\varepsilon) \to \Phi_i \nabla y \varphi_i \), strongly in \( L^2(\Omega \times Y_i) \). Hence, we can pass to the limit and we obtain:
\[
\int_{\Omega \times Y_1} A_1(\nabla u + \nabla y \hat{\nu}) \cdot \nabla y \varphi_1 \Phi_1 \, dx dy + \\
\int_{\Omega \times Y_2} A_2(\nabla v + \nabla y \tilde{\nu}) \cdot \nabla y \varphi_2 \Phi_2 \, dx dy = 0.
\]
(12)

Putting together (11) and (12) and using standard density arguments, we get:
\[
\int_{\Omega \times Y_1} A_1(\nabla u + \nabla y \hat{\nu}) \cdot (\nabla \Phi_1 + \nabla y \tilde{\varphi}_1) \, dx dy + \\
\int_{\Omega \times Y_2} A_2(\nabla v + \nabla y \tilde{\nu}) \cdot (\nabla \Phi_2 + \nabla y \tilde{\varphi}_2) \, dx dy + \\
\int_{\Omega \times Y_1} \alpha(u) \Phi_1 \, dx dy + \int_{\Omega \times Y_2} \beta(v) \Phi_2 \, dx dy + H \int_{\Omega} (v - u)(\Phi_2 - \Phi_1) \, dx = \\
\int_{\Omega \times Y_1} f \Phi_1 \, dx dy + \int_{\Omega \times Y_2} f \Phi_2 \, dx dy,
\]
(13)
for \( \Phi_1, \Phi_2 \in H^1_0(\Omega) \), \( \tilde{\varphi}_1 \in L^2(\Omega; H^1_{\text{per}}(Y_1)) \) and \( \tilde{\varphi}_2 \in L^2(\Omega; H^1_{\text{per}}(Y_2)) \).

So, we have the variational formulation of the limit problem (5). Since \( u \) and \( v \) are uniquely determined (see [14] and [18]), the whole sequences of microscopic solutions converge to a solution of the unfolded limit problem and this completes the proof of Theorem 1.

Let us discuss now briefly the other two interesting cases. We notice that
\[
\| T_1^\varepsilon(u^\varepsilon) - T_2^\varepsilon(v^\varepsilon) \|_{L^2(\Omega \times Y)} \leq C \varepsilon^{\frac{1}{12}}.
\]
Therefore, for the case $\gamma = 0$, we have, at the macroscale, $u = v = u^0$ and, by unfolding, we get

\[
\int_{\Omega \times Y_1} A_1 (\nabla u^0 + \nabla_y \hat{u}) \cdot (\nabla_x \Phi + \nabla_y \tilde{\varphi}_1) \, dx \, dy + \\
\int_{\Omega \times Y_2} A_2 (\nabla u^0 + \nabla_y \hat{v}) \cdot (\nabla_x \Phi + \nabla_y \tilde{\varphi}_2) \, dx \, dy + \\
\int_{\Omega \times Y_1} \alpha(u^0) \Phi \, dx \, dy + \int_{\Omega \times Y_2} \beta(u^0) \Phi \, dx \, dy = \\
\int_{\Omega \times Y_1} f \Phi \, dx \, dy + \int_{\Omega \times Y_2} f \Phi \, dx \, dy, \quad (14)
\]

for $\Phi \in H^1_0(\Omega)$, $\tilde{\varphi}_1 \in L^2(\Omega; H^1_{per}(Y_1))$ and $\tilde{\varphi}_2 \in L^2(\Omega; H^1_{per}(Y_2))$, which leads immediately to the macroscopic problem (6).

For the case $\gamma = -1$, we also obtain at the macroscale $u = v = u_0$. Moreover, we can prove that

\[
\frac{T_1^\varepsilon(u^\varepsilon) - T_2^\varepsilon(u^\varepsilon)}{\varepsilon} \to \hat{u} - \hat{v} \quad \text{weakly in } L^2(\Omega \times \Gamma).
\]

Hence, by unfolding, we get

\[
\int_{\Omega \times Y_1} A_1 (\nabla u_0 + \nabla_y \hat{u}) \cdot (\nabla_x \Phi + \nabla_y \tilde{\varphi}_1) \, dx \, dy + \\
\int_{\Omega \times Y_2} A_2 (\nabla u_0 + \nabla_y \hat{v}) \cdot (\nabla_x \Phi + \nabla_y \tilde{\varphi}_2) \, dx \, dy + \\
\int_{\Omega \times Y_1} \alpha(u_0) \Phi \, dx \, dy + \int_{\Omega \times Y_2} \beta(u_0) \Phi \, dx \, dy + \int_{\Omega \times \Gamma} h_0(\hat{v} - \hat{u})(\tilde{\varphi}_2 - \tilde{\varphi}_1) \, dx \, d\sigma = \\
\int_{\Omega \times Y_1} f \Phi \, dx \, dy + \int_{\Omega \times Y_2} f \Phi \, dx \, dy, \quad (15)
\]

which gives exactly the limit problem (7).

5. CONCLUSIONS

Using the recently developed periodic unfolding method, the macroscopic behavior of the solution of a problem describing the heat transfer in a periodic composite material formed by two constituents, separated by an imperfect interface, was analyzed. One advantage offered by our approach is that it allows us to avoid the use of extension operators and, as a result, to deal with much more general media. Our setting is also relevant for studying the electrical conduction in biological tissues.
REFERENCES