QUANTUM AND CLASSICAL LIE SYSTEMS FOR EXTENDED SYMPLECTIC GROUPS

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In the framework of Lie systems, we study the affine symplectic group $G^{AS}$ and the Jacobi group $G^{J}$. We construct canonical bases for the irreducible unitary representations of $G^{J}$ consisting of $K^{J}$-vectors, where $K^{J}$ is the maximal compact subgroup of $G^{J}$. We study the quantum Lie systems based on the extended Poincaré disk $\mathbb{D}$ diffeomorphic to the maximal elliptic coadjoint orbit of $G^{J}$. We establish the quasienergy operator reduced to $\mathbb{D}$ and the corresponding Wei-Norman equations. Moreover, we obtain the solutions of the time-dependent Schrödinger equation with the initial states represented by $K^{J}$-vectors. Finally, we obtain a Poisson algebra isomorphism between the quantum and classical Lie systems based on the Poisson manifold $\mathbb{D}$.

Key words: Lie system, affine symplectic group, Jacobi group, extended Poincaré disk, quasienergy operator, Schrödinger equation.

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1. INTRODUCTION

Some unitary representations of extended symplectic groups [1]–[3] have been investigated in the framework of geometric quantization [4], [5], nuclear collective models [6]–[10], geometry of collective models [4], [11], quantum dynamics of trapped ions [12]–[14], quantum mechanics [15]–[18], signal processing [19]. The theory of Lie systems of differential equations [20], [21] involves many problems in physics and in mathematics: Wei-Norman equations, differential geometry on Hilbert spaces, exactly solvable Schrödinger equations, integrability conditions for time-dependent harmonic oscillators, Ermakov systems, Milne–Pinney equations, high-order Riccati equations, stochastic Lie–Scheffers systems, Lie–Hamilton systems, Lie superalgebras, differential Galois theory, automorphic systems [22]–[27] (and references therein).

This paper is organized as follows. In Section 2 we review briefly some basic facts about the affine symplectic group $G^{AS} = \text{SL}(2, \mathbb{R}) \rtimes \mathbb{R}^{2}$ and the Jacobi group $G^{J} = \text{SL}(2, \mathbb{R}) \rtimes \mathbb{H}_{3}(\mathbb{R})$, where $\mathbb{H}_{3}(\mathbb{R})$ is the three-dimensional Heisenberg group. We construct explicit canonical bases for the irreducible unitary representations of
$G^J$ consisting of $K^J$-vectors, where $K^J$ is the maximal compact subgroup of $G^J$ (Proposition 2.2). In Section 3 we study the quantum Lie systems based on the extended Poincaré disk $\mathcal{M} = \mathbb{D} \times \mathbb{R}^2$, where $\mathbb{D}$ is the open unit disk. We obtain the quasienergy operator reduced to $\mathcal{M}$ and explicit expression of the corresponding Weigner equations (Proposition 3.1). Moreover, we obtain the solutions to the time-dependent Schrödinger equations with the initial states represented by $K^J$-vectors (Corollary 3.2). In Section 4 we study the classical Lie systems based on the Poisson manifold $\mathcal{M}$. We construct a Poisson algebra isomorphism between the quantum and classical Lie systems (Proposition 4.1). Finally, an algebraic dequantization is presented (Proposition 4.2).

Notation. Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the field of real numbers, the field of complex numbers, the ring of integers, and the set of non-negative integers, respectively. $M_n(\mathbb{R})$ denotes the set of $n \times n$ real matrices. The identity matrix of degree $n$ is denoted by $I_n$. The unit operator is denoted by $I$. Let $\langle X_1, \ldots, X_n \rangle_\mathbb{R}$ denote the real Lie algebra with the basis elements $X_1, \ldots, X_n$.

2. UNITARY REPRESENTATIONS OF EXTENDED SYMPLECTIC GROUPS

The symplectic group $\text{Sp}(1, \mathbb{R})$ is identified with the group $\text{SL}(2, \mathbb{R})$ of all matrices $\sigma \in M_2(\mathbb{R})$ with $\det \sigma = 1$. The Jacobi group $G^J = \text{SL}(2, \mathbb{R}) \ltimes H_3(\mathbb{R})$ is the semidirect product of the group $\text{SL}(2, \mathbb{R})$ with the Heisenberg group $H_3(\mathbb{R})$ endowed with the following multiplication law [2]:

$$\begin{align*}
(\sigma, (\lambda_1, \lambda_2, \varpi)) \cdot (\sigma', (\lambda_1', \lambda_2', \varpi')) &= (\sigma \sigma', (\lambda_1 + \lambda_1', \lambda_2 + \lambda_2', \varpi')), \\
\varpi'' &= \varpi + \varpi' + \lambda_1 \lambda_2' - \lambda_2 \lambda_1',
\end{align*}
$$

where $(\lambda_1, \lambda_2, \varpi), (\lambda_1', \lambda_2', \varpi') \in \mathbb{R}^3$ and $\sigma, \sigma' \in \text{SL}(2, \mathbb{R})$. The Heisenberg group $H_3(\mathbb{R})$ consists of all $g = (0, (\lambda_1, \lambda_2, \varpi)) \in G^J$. The affine symplectic group $G^{AS} = \text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ consists of all elements $g = (\sigma, (\lambda, \mu, 0)) \in G^J$. The center $Z^J \cong \mathbb{R}$ of $G^J$ consists of all $g = (0, (0, 0, \varpi)) \in G^J$. Then $G^J$ is a central extension of $G^{AS}$.

We now consider the matrices $E_{ij} \in M_4(\mathbb{R})$, $1 \leq i, j \leq 4$, defined by $(E_{ij})_{\alpha\beta} = \delta_{i\alpha} \delta_{j\beta}$, $1 \leq \alpha, \beta \leq 4$. We introduce the matrices [2]:

$$\begin{align*}
P &= E_{21} - E_{34}, Q = E_{14} - E_{23}, R = E_{24}, \\
F &= E_{13}, G = E_{31}, H = E_{11} - E_{33}.
\end{align*}
$$

We get the commutators

$$\begin{align*}
\end{align*}
$$

all other are zero [2]. The Lie algebras of $G^J$, $H(\mathbb{R})$, $G^{AS}$, and $\text{SL}(2, \mathbb{R})$ are denoted by $g^J$, $\mathfrak{h}$, and $\mathfrak{sl}(2, \mathbb{R})$, respectively. Then
$g^J = \langle P, Q, R, F, G, H \rangle_\mathbb{R}, \ h = \langle P, Q, R \rangle_\mathbb{R}, \ \mathfrak{sl}(2, \mathbb{R}) = \langle F, G, H \rangle_\mathbb{R}.$ \hspace{1cm} (4)

Every non-trivial central character $\psi$ of index $m_0 \in \mathbb{R}$ can be obtained as $\psi((0, 0, \varepsilon)) = \exp(2\pi i m_0 \varepsilon)$, where $(0, 0, \varepsilon) \in H_2(\mathbb{R})$. We will concentrate throughout on representations having positive central character.

Let $\pi$ a unitary irreducible representation of $G^J$ of positive index $m_0$ on a complex separable Hilbert space $\mathcal{H}_\pi$. Let $d\pi$ be the derived representation of $\pi$. Denote $X = d\pi(X)$, where $X \in g^J$. Then $R = i\mu I$, where $\mu = 2\pi m_0$ and $I$ is the identity operator. We introduce the following operators:

$$K_0 = \frac{i}{2}(G - F), \ K_1 = \frac{i}{2}(G + F), \ K_2 = \frac{i}{2}H,$$ \hspace{1cm} (5)

$$N_1 = -i\mu^{-1/2}Q, \ N_2 = i\mu^{-1/2}P.$$ \hspace{1cm} (6)

By (3), (5), and (6), we obtain the commutators

$$[N_2, K_1] = [K_2, N_2] = [N_2, K_0] = \frac{i}{2}N_1,$$ \hspace{1cm} (7)

$$[N_1, K_1] = [N_1, K_2] = [K_0, N_1] = \frac{i}{2}N_2,$$ \hspace{1cm} (8)

$$[N_2, N_1] = 2iI, \ [K_0, K_1] = iK_2, \ [K_2, K_0] = iK_1, \ [K_2, K_1] = iK_0.$$ \hspace{1cm} (9)

Let $Mp(2, \mathbb{R})$ be the non-trivial two-fold cover of $SL(2, \mathbb{R})$. A classification of irreducible unitary representations of the Jacobi group $G^J$ is given by the following theorem [2], [3]:

**Theorem 2.1.** (i) Any representation $\pi$ of $G^J$ with index $m \neq 0$ is obtained in a unique way as $\pi = \pi^m_{SW} \otimes \pi'$, where $\pi^m_{SW}$ is the Schrödinger-Weil representation and $\pi'$ is a representation of the metaplectic group $Mp(2, \mathbb{R})$. The representations $\pi$ and $\pi'$ are simultaneously unitary and irreducible.

(ii) Any irreducible unitary representation $\pi$ of $G^J$ of index $m \neq 0$ is infinitesimally equivalent to one of the following representations:

(a) a principal series representation $d\pi_{m,s,\nu} = d\pi^m_{SW} \otimes d\pi^{[s,\nu]}$ for $\nu = \pm 1/2, \ s \in i\mathbb{R} \cup (-1/2, 1/2)$;

(b) a positive discrete series representation $d\pi^+_{m,k} = d\pi^m_{SW} \otimes d\pi^{[k_0,+]}$ for $k = k_0 + 1/2 \in \mathbb{Z}, \ k \geq 1$;

(c) a negative discrete series representation $d\pi^-_{m,k} = d\pi^m_{SW} \otimes d\pi^{[k_0,-]}$ for $k = k_0 + 1/2 \in \mathbb{Z}, \ k \geq 1$.

The only equivalence between these representations are $d\pi_{m,s,\nu} \simeq d\pi_{m,-s,\nu}$.

$K_0$ has the integral dominants weights $k/2$ for $d\pi^+_{m,k}$ and $(1-k)/2$ for $d\pi^-_{m,k}$.

A canonical basis of $\mathcal{H}_\pi$ is an orthonormal basis of eigenvectors of $K_0$. 


\textbf{Proposition 2.2.} There exist the canonical bases
\begin{equation}
\mathfrak{B}^{[m,k,\pm]} = \left\{ \phi_{nj}^{[m,k,\pm]} \mid n, j \in \mathbb{N} \right\}, \mathfrak{B}^{[m,s,\nu]} = \left\{ \phi_{nj}^{m,s,\nu} \mid n \in \mathbb{N}, j \in \mathbb{Z} \right\}
\end{equation}
for the representation spaces of $d\pi_{m,k}^\pm$ and $d\pi_{m,s,\nu}$ such that the following relations hold:
\begin{equation}
K_0 \phi_{nj}^{[m,s,\nu]} = \left( j + \frac{n + \nu}{2} + \frac{1}{4} \right) \phi_{nj}^{[m,s,\nu]},
\end{equation}
\begin{equation}
K_0 \phi_{nj}^{[m,k,\pm]} = \left( \pm j + \frac{n \pm k + 1}{2} \right) \phi_{nj}^{[m,k,\pm]}.
\end{equation}
\textbf{Proof.} Let $\pi_{SW}^m$ be the Schrödinger-Weil representation of the Jacobi group $G^J$ on the Hilbert space $L^2(\mathbb{R})$. The basis of the Lie algebra $d\pi_{SW}^m(g^I)$ is given by the following differential operators [2]:
\begin{equation}
P_{SW} = \frac{d}{dx}, Q_{SW} = 2i\mu x, R_{SW} = i\mu I,
\end{equation}
\begin{equation}
F_{SW} = i\mu x^2, G_{SW} = \frac{i}{\mu} \frac{d^2}{dx^2}, H_{SW} = x \frac{d}{dx} + \frac{1}{2} I.
\end{equation}
We introduce the operators
\begin{equation}
b = i\sqrt{\mu x} + \frac{1}{2\sqrt{\mu}} \frac{d}{dx}, b^\dagger = i\sqrt{\mu x} + \frac{1}{2\sqrt{\mu}} K_{0SW} = \frac{1}{2} b^\dagger b + \frac{1}{4} I.
\end{equation}
The Fock canonical basis $\mathfrak{B}_{SW}$ of $L^2(\mathbb{R})$ consists of the vectors $\varphi_n, n \in \mathbb{N}$, where $b\varphi_0 = 0, ||\varphi_0|| = 1$, and $\varphi_n = (nt)^{-1/2}(b^\dagger)^n\varphi_0$. Consider the orthonormalized bases $B^{[k,\pm]} = \left\{ \phi_{nj}^{[k,\pm]} \mid j \in \mathbb{N} \right\}$ and $B^{[s,\nu]} = \left\{ \phi_{nj}^{[s,\nu]} \mid j \in \mathbb{Z} \right\}$ for the Waldspurger representations $\pi_{[s,\nu]}$ and $\pi_{[k,\pm]}$, where $k = k_0 + 1/2$, such that [28], [2]:
\begin{equation}
K_0' \phi_{nj}^{[s,\nu]} = \left( j + \frac{\nu}{2} \right) \phi_{nj}^{[s,\nu]}, K_0' \phi_{nj}^{[m,k,\pm]} = \left( \pm j + \frac{k}{2} + \frac{1}{4} \right) \phi_{nj}^{[m,k,\pm]}.
\end{equation}
Here $K_0 = K_{0SW} \otimes K_0'$. Let $\phi_{nj}^{[m,k,\pm]} = \varphi_m \otimes \psi_j^{[k,\pm]}, \phi_{nj}^{[m,s,\nu]} = \varphi_m \otimes \psi_j^{[s,\nu]}, \phi_{nj}^{[m,k,\pm]} = \varphi_m \otimes \psi_j^{[k,\pm]}$. By (13) and (14), we obtain (11).

The irreducible unitary representations of $G^{AS}$ are the following [2]:
1) The irreducible unitary representations $\pi$, where the restriction of $\pi$ to $\mathbb{R}^2$ is trivial and the restriction of $\pi$ to $\text{SL}(2, \mathbb{R})$ is an irreducible unitary representation of $\text{SL}(2, \mathbb{R})$.
2) The representations $\pi_r = \text{Ind}_{G_0}^G \tau_r$ induced from the unitary character $\tau_r$ of $G_0$ defined by
\begin{equation}
\pi_r((\sigma_c, (\lambda_1, \lambda_1, 0))) = \exp(2\pi i (rc + \lambda_1)), \sigma_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, r, c, \lambda_1, \lambda_2 \in \mathbb{R},
\end{equation}
where $G_0$ consists of all $(\sigma, (\lambda_1, \lambda_2, 0) \in G^{as}$ and $r, c, \lambda_1, \lambda_2 \in \mathbb{R}$.

For the canonical bases of the irreducible unitary representations of $\text{SL}(2, \mathbb{R})$ we refer to [29], [30].

### 3. Quantum Lie Systems Based on the Extended Poincaré Disk

Let $\pi$ be a unitary irreducible representation of the Jacobi group $G^J$ on the Hilbert space $\mathcal{H}_\pi$. The time evolution of a quantum mechanical system with time-dependent Hamiltonian is determined by the time-dependent Schrödinger equation

$$i\hbar \frac{d\psi(t)}{dt} = \mathbf{H}(t)\psi(t),$$

where $\mathbf{H}(t)$ is a self-adjoint operator on the Hilbert space $\mathcal{H}_\pi$ and $\psi(t) \in \mathcal{H}_\pi$ for every $t \in J$. Here $J \subset \mathbb{R}$ is an open interval with $0 \in J$. Let $\mathcal{V}_J$ be the set of functions $f : J \to \mathbb{R}$ summable on every finite interval. We shall normally not indicate the time dependence. We consider the quantum Lie system described by

$$\mathbf{H}_0 = 2\varepsilon_0 \mathbf{K}_0 + 2\varepsilon_1 \mathbf{K}_1 + 2\varepsilon_2 \mathbf{K}_2 + \nu_1 \mathbf{N}_1 + \nu_2 \mathbf{N}_2,$$

where $\varepsilon_0, \varepsilon_1, \varepsilon_2, \nu_1, \nu_2 \in \mathcal{V}_J$. The Poincaré disk $\mathfrak{D}$ consists of all points $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 < 1$. The extended Poincaré disk $\mathfrak{M}$ consists of all points $\zeta = (u, v, x, y) \in \mathbb{R}^4$, where $(u, v) \in \mathfrak{D}$ and $(x, y) \in \mathbb{R}^2$. The real analytic manifold $\mathfrak{M}$ can be identified with the homogeneous spaces $G^J/\mathbb{Z}\times K^J$ and $G^{AS}/K^{AS}$, where $K^{AS} \cong \text{SO}(2)$ is the maximal compact subgroup of $G^{AS}, K^J \cong \text{SO}(2)$ is the maximal compact subgroup of $G^J$, and $\mathbb{Z} \cong \mathbb{R}$ is the center of $G^J$. Consider now the section $T : \mathfrak{M} \to \pi(G^J)$ given by the unitary operators

$$T(\zeta) = D(x, y)S(u, v), \quad \zeta = (u, v, x, y) \in \mathfrak{M},$$

$$D(x, y) = \exp(\imath y \mathbf{N}_1 + \imath x \mathbf{N}_2), \quad S(u, v) = \exp(\imath k_1 \mathbf{K}_1 + \imath k_2 \mathbf{K}_2),$$

$$k_1 = \frac{v}{2s} \ln \frac{1 + s}{1 - s}, \quad k_2 = \frac{u}{2s} \ln \frac{1 + s}{1 - s}, \quad s = (u^2 + v^2)^{1/2}.$$

For any linear operator $\mathbf{A}$ and $\zeta \in \mathfrak{M}$ define the operators

$$\mathbf{A}(\zeta) = T^{-1}(\zeta) \mathbf{A}T(\zeta).$$

We now introduce the following family of unitary operators:

$$U(\zeta, \varphi) = \exp(-\imath \varphi)T(\zeta),$$

where $\zeta \in \mathfrak{M}$ and $\varphi$ is a real phase. Let $\tau = t/\hbar$. Consider the quasienergy operator $\mathbf{E} = \imath d/d\tau - \mathbf{H}$ acting in some enlarged Hilbert space $[31]$. Then

$$\mathbf{E}(\zeta, \varphi) = U(-\zeta, -\varphi)\mathbf{E}U(\zeta, \varphi) = \frac{d\varphi}{d\tau} \mathbf{I} + T(\zeta)^{-1} \frac{dT(\zeta)}{d\tau} - \mathbf{H}(\zeta).$$

This completes the proof.
Proposition 3.1. Suppose \( H = H_0 + V \). Then
\[
E(\zeta, \varphi) = \left( \frac{d\varphi}{d\tau} - 2\nu_1 x + 2\nu_2 y \right) I - 2(\varepsilon_0 + \varepsilon_1 u - \varepsilon_2 v) K_0 - V(\zeta),
\]
where \( \zeta = (u, v, x, y) \in \mathfrak{M} \) and the functions \( u, v, x, y, \varphi : J \to \mathbb{R} \) satisfy the differential equations
\[
\begin{align*}
\frac{du}{d\tau} &= 2v(\varepsilon_1 u + \varepsilon_0) - \varepsilon_2 (1 - u^2 + v^2), \\
\frac{dv}{d\tau} &= 2u(\varepsilon_2 v - \varepsilon_0) - \varepsilon_1 (1 + u^2 - v^2), \\
\frac{dx}{d\tau} &= -\varepsilon_2 x + (\varepsilon_0 - \varepsilon_1) y - \nu_2, \\
\frac{dy}{d\tau} &= -(\varepsilon_0 + \varepsilon_1) x + \varepsilon_2 y - \nu_1.
\end{align*}
\]

Proof. For any \( X \in d\pi \), let \( \text{ad}X : d\pi \to d\pi \) denote the linear operator defined by
\[
(\text{ad}X)Y = [X, Y], \quad (\text{ad}X)^n Y = [X, (\text{ad}X)^{n-1} Y], \quad n \in \mathbb{N} \setminus \{0\},
\]
where \( Y \in d\pi \). The Baker-Hausdorff formula can be written as [22]
\[
(\exp X)Y \exp(-X) = Y + \sum_{n=1}^{\infty} \frac{1}{n!} (\text{ad}X)^n Y,
\]
where \( X, Y \in d\pi(g) \). It is convenient to denote
\[
S_{uv} A = S(-u, -v) A S(u, v), \quad D_{xy} A = D(-x, -y) A D(x, y),
\]
where \( A \in i d\pi(g) \) and \( \zeta = (u, v, x, y) \in \mathfrak{M} \). From the commutation relations (7) and (9), one derives
\[
N_1(\zeta) = r(1 + u) N_1 - r v N_2 + 2x I, \quad N_2(\zeta) = r(1 - u) N_1 - r v N_1 - 2y I,
\]
\[
\begin{align*}
D_{xy} K_0 &= K_0 + \frac{x}{2} N_1 - \frac{y}{2} N_2 + \frac{x^2 + y^2}{2} I, \\
D_{xy} K_1 &= K_1 + \frac{x}{2} N_1 + \frac{y}{2} N_2 + \frac{x^2 - y^2}{2} I, \\
D_{xy} K_2 &= K_2 + \frac{x}{2} N_2 - \frac{y}{2} N_1 - xy I.
\end{align*}
\]
\[
\begin{align*}
S_{uv} K_0 &= r^2 (1 + u^2 + v^2) K_0 + 2r^2 u K_1 - 2r^2 v K_2, \\
S_{uv} K_1 &= r^2 (1 + u^2 - v^2) K_1 + 2r^2 u K_0 - 2r^2 u v K_2, \\
S_{uv} K_2 &= r^2 (1 - u^2 + v^2) K_2 - 2r^2 v K_0 - 2r^2 u v K_1,
\end{align*}
\]
\[
-iT^{-1}(\zeta) \frac{dT(\zeta)}{d\tau} = 2r^2 \left( \frac{dv}{d\tau} - v \frac{du}{d\tau} \right) K_0 + 2r^2 \frac{dK_1}{d\tau} + 2r^2 \frac{dK_2}{d\tau}
\]
\[
+ r \left( \frac{dy}{d\tau} + u \frac{dy}{d\tau} - v \frac{dx}{d\tau} \right) N_1 + r \left( \frac{dx}{d\tau} - u \frac{dx}{d\tau} - v \frac{dy}{d\tau} \right) N_2 + \left( \frac{dx}{d\tau} - y \frac{dx}{d\tau} \right) I.
\]
The explicit form of \( H_0(\zeta) = D_{xy}S_{uv}H_0 \) is determined by (17), (32)-(38). Then (23), (25)-(28), and (39) imply (24). We remark that (25)-(28) are equations of Wei-Norman type on \( \mathfrak{N} \). Equations (25) and (26) are equivalent to a complex Riccati equation obtained in [12]-[14]. The Wei-Norman equations for the Jacobi group \( G^J \) have been established in [26] and [27]. Moreover, some solutions of these equations have been given in [32], [33] (and references therein).

**Corollary 3.2.** Let \( \Phi \in \mathcal{H} \) be an eigenvector of \( K_0 \) with the eigenvalue \( \kappa \in \mathbb{R} \). Assume that the differential equations (25)-(26) and

\[
\frac{d\varphi}{dt} = 2(\nu_1 x - \nu_2 y) + 2\kappa(\varepsilon_0 + \varepsilon_1 u - \varepsilon_2 v)
\]

are satisfied. Then \( \Psi(\zeta, \varphi) = U(\zeta, \varphi)\Phi \), where \( \zeta = (u, v, x, y) \), is a solution of the time-dependent Schrödinger equation (16) for \( H = H_0 \).

The spectrum of \( K_0 \) is described by Proposition 2.2. Equations (25) and (26) are equivalent to

\[
\frac{dW}{d\tau} = -(\varepsilon_0 + \varepsilon_1)W^2 - 2\varepsilon_2 W - \varepsilon_0 + \varepsilon_1,
\]

where \( W = (1 + u + iv)(v + i - iu)^{-1} \) and \( \text{Im}W > 0 \). The differential equation (41) is linear for \( \varepsilon_0 + \varepsilon_1 = 0 \). If \( \varepsilon_0 + \varepsilon_1 \neq 0 \), then the Riccati equation (41) can be linearized by substituting \( W = -(aZ)^{-1}dZ/d\tau \), where \( Z \) satisfies the linear differential equation

\[
\frac{dZ}{d\tau^2} + 2\varepsilon_2 \frac{dZ}{d\tau} + \varepsilon_0 - \varepsilon_1^2 = 0
\]

The solutions of (25),(26), and (42) are systematically described in the general theory of systems of linear differential equations [34]. If \( \varepsilon_0 \) and \( \varepsilon_1 \) are time–periodic functions and \( \varepsilon_2 = 0 \), then is a Hill equation [34]. Using the stable solutions of this equation, some discrete quasienergy spectra and the corresponding quasienergy states for charged particles in the Paul trap have been explicitly obtained in [12]-[14].

### 4. Classical Lie Systems Based on the Extended Poincaré Disk

The homogeneous space \( \mathfrak{N} \) is naturally diffeomorphic to an elliptic coadjoint orbit of the Jacobi group \( G^J \). \( \mathfrak{N} \) is a symplectic manifold with respect to the Kirillov–Kostant form given in [3]. We refer to [35] for a detailed discussion on symplectic and Poisson manifolds.

Consider two positive real numbers \( \alpha \) and \( \beta \). The extended Poincaré disk \( \mathfrak{N} \) is a Poisson manifold with the Poisson bracket \( \{,\} \) defined as

\[
\{f,g\} = \frac{1}{\alpha^2} \left( \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} - \frac{\partial f}{\partial v} \frac{\partial g}{\partial u} \right) + \frac{1}{\beta} \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right),
\]

(43)
where \( f, g \in C^\infty(\mathcal{M}) \), \((u, v, x, y) \in \mathcal{M}\), and \( r = (1 - u^2 - v^2)^{-1/2}\).

We introduce the functions \( \mathcal{N}_j, \mathcal{K}_j \in C^\infty(\mathcal{M}) \), where \( j = 0, 1, 2 \), defined as

\[
\mathcal{N}_1(\zeta) = \beta x, \quad \mathcal{N}_2(\zeta) = -\beta y, \quad \mathcal{N}_0(\zeta) = \beta, \quad (44)
\]

\[
\mathcal{K}_0(\zeta) = \frac{\alpha}{4} (2r^2 - 1) + \frac{\beta}{4} (x^2 + y^2), \quad (45)
\]

\[
\mathcal{K}_1(\zeta) = \frac{\alpha}{2} r^2 u + \frac{\beta}{4} (x^2 - y^2), \quad \mathcal{K}_2(\zeta) = -\frac{\alpha}{2} r^2 v - \frac{\beta}{2} xy. \quad (46)
\]

By (43)-(46), we obtain the commutation relations

\[
\{\mathcal{K}_0, \mathcal{K}_1\} = \mathcal{K}_2, \quad \{\mathcal{K}_2, \mathcal{K}_0\} = \mathcal{K}_1, \quad \{\mathcal{K}_2, \mathcal{K}_1\} = \mathcal{K}_0, \quad \{\mathcal{N}_2, \mathcal{N}_1\} = 2\mathcal{N}_0, \quad (47)
\]

\[
\{\mathcal{N}_2, \mathcal{K}_1\} = \{\mathcal{K}_2, \mathcal{N}_0\} = \{\mathcal{N}_2, \mathcal{K}_0\} = \frac{1}{2}\mathcal{N}_1, \quad (48)
\]

\[
\{\mathcal{N}_1, \mathcal{K}_1\} = \{\mathcal{N}_1, \mathcal{K}_2\} = \{\mathcal{K}_0, \mathcal{N}_1\} = \frac{1}{2}\mathcal{N}_2. \quad (49)
\]

We now consider a classical Lie system described by the Hamiltonian \( \mathcal{F} = 2\varepsilon_0 \mathcal{K}_0 + 2\varepsilon_1 \mathcal{K}_1 + 2\varepsilon_2 \mathcal{K}_2 + \nu_1 \mathcal{N}_1 + \nu_2 \mathcal{N}_2 \), \( (50) \)

where \( \varepsilon_0, \varepsilon_1, \varepsilon_2, \nu_1, \nu_2 \in \mathcal{V}_J \).

Let \( A_{\alpha, \beta} \) be the Poisson algebra with the basis consisting of the functions \( \mathcal{N}_j \) and \( \mathcal{K}_j \), where \( j = 0, 1, 2 \). Let \( A_{\pi} \) be the Poisson algebra of all operators \( A = iX \), where \( X \in d\pi(g) \), with the bracket \( \{\cdot, \cdot\}' \) defined by \( \{A, B\}' = -i[A, B] \) for any \( A, B \in A_{\pi} \). We introduce the map \( \rho : A_{\pi} \to A_{\alpha, \beta} \) defined by \( \rho(K_j) = K_j, \rho(N_j) = N_j \), where \( j = 0, 1, 2 \).

**Proposition 4.1.** \( \rho \) is an isomorphism of Poisson algebras. The equations of motion for \( \rho(H_0) = \mathcal{F} \) are precisely the equations of Wei-Norman type given by (25)-(28).

**Proof.** The equations of motion in Poisson bracket form \( df / d\tau = \{f, \mathcal{F}\} \), for \( f = u, v, x, y \), can be written as

\[
\frac{du}{d\tau} = \frac{1}{\alpha r^4} \frac{\partial \mathcal{F}}{\partial v}, \quad \frac{dv}{d\tau} = -\frac{1}{\alpha r^4} \frac{\partial \mathcal{F}}{\partial u}, \quad \frac{dx}{d\tau} = \beta \frac{\partial \mathcal{F}}{\partial y}, \quad \frac{dy}{d\tau} = -\frac{1}{\beta} \frac{\partial \mathcal{F}}{\partial x}. \quad (51)
\]

The equations (51) and (44)-(46) imply (25)-(28).

We now present an explicit dequantization on \( \mathcal{M} \). Assume that \( \Phi \in \mathcal{H}_\pi \) is an eigenvector of \( K_0 \) with the positive eigenvalue \( \kappa \). Consider the map \( \delta : A_{\pi} \to C^\infty(\mathcal{M}) \) defined by \( \delta(A) = \hat{A} \) where \( \hat{A}(\zeta) = \langle \Phi, A(\zeta) \Phi \rangle \) for any \( \zeta \in \mathcal{M} \).

**Proposition 4.2.** There is a Poisson algebra isomorphism \( \rho : A_{\pi} \to A_{\kappa, 1} \) such that \( \delta(A) = \rho(A) \) for any \( A \in A_{\pi} \).

**Proof.** By (32)-(38), we have \( \delta(K_j) = K_j \) and \( \delta(N_j) = N_j \) for \( j = 0, 1, 2 \).
The dequantization $\delta$ can be extended to polynomial Hamiltonians in $K_j$ and $N_j$, where $j = 0, 1, 2$. According to (32)-(38) the dequantized Hamiltonians are polynomial functions in $K_j$ and $N_j$, where $j = 0, 1, 2$.

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