In this work we analytically derived the critical behavior of zero-temperature quantum phase transition for two generalized Dicke models: one model includes quadratic field terms (the $A^2$-term) and the other one has dipole-dipole interaction. It was pointed out that both the $A^2$-term and dipole-dipole interaction lead to a corrected position of the critical point, amounting to inducing a shift of radiation field frequency and atomic transition frequency.

Key words: generalized Dicke model, quantum transition, spin-coherent-state.

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1. INTRODUCTION

The light-matter interaction has been the central topic of quantum optics in view that it is the basis for laser theory, quantum state engineering, fundamental testing of quantum mechanics, and implementation of quantum information processing. A two-level atom interacting with a radiation field is the building block among the various systems involving the interaction of photons and atoms [1]. Dicke model (DM) containing $N$ two-level atoms interacting with a radiation field mode was firstly used to illustrate the importance of collective effects of superradiance, where the atomic ensemble spontaneously emits with an intensity proportional to $N^2$ rather than $N$ [2]. And the Dicke Hamiltonian has been used extensively in theoretical quantum optics for it provides a simple but very fruitful starting point for studies of the collective and cooperative interactions of radiation and matter [3–5]. A thermodynamic phase transition from the normal to the superradiant phase for DM was first investigated in rotating-wave approximation (RWA) by Hepp and Lieb [6]. Wang and Hioe provided a much simplified treatment by making use of Glauber’s coherent state for the field [7] and then considered the generalized DM without RWA [8]. These researches conclude that when the atom-field coupling constant $\lambda$ is below the critical value $\lambda_c$, no phase transition occurs for any temperature, whereas when $\lambda$ is above $\lambda_c$ there exists a phase transition at a finite temperature $T_c$. Above this critical temperature $T_c$, the system is in the normal phase, whereas for $T < T_c$, the system is in
the so called superradiant phase. Later Emary and Brandes demonstrated the quantum phase transition (QPT) for DM without RWA using the Holstein-Primakoff (HP) series expansion of the Dicke Hamiltonian truncated to second order in terms of the ratio between the number of excited atoms to the total number of atoms \cite{9, 10}. The results show DM exhibits a QPT at a critical coupling $\lambda_{c}$, where the ground state changes from an unexcited normal phase to a symmetry-broken superradiant phase in which both the field and atomic collection acquire macroscopic occupations.

It is well known that the Dicke Hamiltonian was constructed under the following assumptions: the quadratic term of vector potential in the interaction of radiation with matter, i.e., the $A^2$-term is negligibly small \cite{3}; dipole-dipole interaction (DDI) is negligible based on the assumption that the distance between particles is small compared with a radiation wavelength but large compared with a particle wavelength \cite{2}. The $A^2$-term represents the energy of mutual interaction between radiation oscillators of the radiation field \cite{3}. Rzewski et al. \cite{11, 12} showed that the classic phase transition disappears when the $A^2$-term is also included in the Dicke Hamiltonian. It is also worth mentioning that in Ref. \cite{13} a simple Green-function method was introduced in order to study the critical behavior of the Dicke model with external fields. And within current solid-state and atomic communities the neglect of interaction between the spins is not always valid, i.e., the direct interaction between atoms can’t be disregarded, thus the generalized DM with DDI has been considered \cite{14–17}. In this paper we point out that either the $A^2$-term or the DDI have in fact the same effect on zero temperature QPT of the generalized DM, that is, producing a shift of radiation field or atomic transition frequency, therefore resulting in a corrected position of the critical point. In fact the radiation field and the collective atom ensemble can be viewed as two oscillators, therefore the $A^2$-term and the DDI represent the internal interaction for the oscillators.

2. GENERALIZED DICKE MODEL WITH $A^2$-TERM

In this section we derive the critical coupling constant of the zero temperature QPT for the generalized DM with $A^2$-term using the spin-coherent-state representation. The non-relativistic Hamiltonian for the radiation-matter interaction can be written as

$$H = \sum_{i=1}^{N} \left[ \frac{1}{2m} \left( \mathbf{p}_i - \frac{e}{c} \mathbf{A}(\mathbf{r}_i) \right)^2 + V(\mathbf{r}_i) \right] + \hbar \omega a^\dagger a$$

which describes a single atomic electron of mass $m$ in a potential $V(\mathbf{r}_i)$ with momentum $\mathbf{p}$ at the position $\mathbf{r}_i$. The last term $\hbar \omega a^\dagger a$ denotes the energy of the radiation field. Vector potential $\mathbf{A}(\mathbf{r}_i)$ with Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ is used to describe the radiation field. Expanding the vector potential $\mathbf{A}(\mathbf{r}_i)$ in creation and annihilation
operators $a^\dagger(a)$ of radiation field the Hamiltonian (1) can be rewritten as a generalized DM in the long-wavelength limit and two-level approximation

$$H^{(1)} = \omega a^\dagger a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}}(J_- + J_+)(a^\dagger + a) + \varepsilon(a + a^\dagger)^2$$

(2)

where $\omega$ and $\omega_0$ are the frequency of radiation field and the atomic transition frequency respectively, $\lambda = \omega_0 d\sqrt{2\pi \hbar/\omega}$ is the effective atom-field coupling constant with $d$ being a projection of the transition dipole moment on the polarization vector of the field mode, $\rho = N/V$ ($V$ is the volume of the system) being the density of atoms, $\varepsilon = e^2 \pi \hbar/(m\omega)$ denotes the interaction between radiation oscillators. The prefactor $1/\sqrt{N}$ gives a finite free energy per atom in the thermodynamical limit $N \to \infty$. $J_z$ is the atomic relative population operator and $J_{\pm}$ are the collective atomic raising and lowering operators, which satisfy the SU(2) Lie algebra $[J_z, J_{\pm}] = \pm J_{\pm}$, where the collective operators are described in terms of standard Pauli matrices of each two-level atom

$$J_u = \frac{1}{2} \sum_{i=1}^{N} \sigma_u^i, \quad (u = x, y, z)$$

$$J_{\pm} = \frac{1}{2} \sum_{i=1}^{N} (\sigma_z^i \pm i\sigma_y^i)$$

(3)

We use a trial wave function being a tensorial product of spin coherent states (SCS) and a boson coherent state $|\theta, \varphi \rangle \otimes |\alpha \rangle$, where the boson coherent state is defined by

$$a |\alpha \rangle = \alpha |\alpha \rangle$$

(4)

and the SCS, or an arbitrary Dicke state, can be generated by rotating the ground state $|j, -j \rangle$ by the angle $\theta$ about the axis $n = (\sin \varphi, -\cos \varphi, 0)$ with $j = N/2$ being the total pseudo-spin value, i.e.,

$$|\theta, \varphi \rangle = R_{\theta, \varphi} |j, -j \rangle$$

(5)

where $R_{\theta, \varphi} = e^{-i\theta J_z} n = e^{-i\theta(J_x \sin \varphi - J_y \cos \varphi)}$. Thus we have the following eigen equation

$$J \cdot n |\theta, \varphi \rangle = -j |\theta, \varphi \rangle$$

(6)

which allow us to calculate analytically the energy function

$$E_{\pm}(\alpha) = \omega \alpha^2 + (4\varepsilon + \omega)u^2 \pm \frac{N}{2} \sqrt{\omega_0^2 + (4\lambda u/\sqrt{N})^2}$$

(7)

Here we defined the boson coherent state $\alpha = \mu + iv$ with $\mu, v$ being real variables. In order to calculate the critical coupling value we minimize the low-lying energy
with respect to $\mu$ and $v$ and find the following self-consistency equations

$$\frac{|\alpha|^2}{N} = \begin{cases} 0, & \lambda < \lambda_c \\ \frac{\omega_0^2}{16\lambda^2} \left( \frac{4\lambda^4}{\lambda_c^4} - 1 \right), & \lambda > \lambda_c \end{cases} \quad (8)$$

Thus the critical point of QPT for the ground state can be obtained exactly and is given by

$$\lambda_c = \sqrt{\tilde{\omega}_0}/2. \quad (9)$$

with $\tilde{\omega} = (\omega + 4\varepsilon)$, which is in accordance with that derived from the free energy in the thermodynamic limit [7, 11] and has been dealt with in Refs. [18, 19]. The scaled ground state energy $E_g$ may be easily calculated

$$\frac{E_g}{N} = \begin{cases} -\frac{\omega_0^2}{2}, & \lambda < \lambda_c \\ \frac{\tilde{\omega}_0\lambda^2}{10\lambda^2} \left( \frac{\lambda^4}{\lambda_c^4} - 1 \right) - \frac{\omega_0\lambda^2}{2\lambda_c^2}, & \lambda > \lambda_c \end{cases} \quad (10)$$

Fig. 1 – The scaled ground-state energy $E_g/N$ and its second-order derivative (inset) versus the coupling constant $\lambda$ for DM including $A^2$-term with parameters $\omega = \omega_0 = 1$ and $\varepsilon = 0.02$.

One can see that in normal phase the $A^2$-term has no effect on the ground state energy for the radiation field is in a vacuum mode whereas in superradiation phase it contributes partly due to the nonlinear interaction between photons. In Fig. 1 we plotted the scaled ground-state energy $E_g/N$ and its second-order derivative as a function of coupling constant $\lambda$ with $\omega = \omega_0 = 1$ (on scaled resonance) and $\varepsilon = 0.02$. It can be seen that the system undergoes a second-order phase transition. Compared
with the non-RWA critical value $\sqrt{\omega_0}/2$ for the standard DM where $\varepsilon = 0$ [6] one can see that the $A^2$-term induces a correction of the critical value and results in a shift of the radiation field frequency, which is simply a consequence of the fact that there are four terms in the $A^2$-term and each term contributes $\varepsilon$ to the mean field. This is one of our main results.

It is worth to point out that the including of $A^2$-term in Dicke Hamiltonian prevents a classical superradiant phase transition [11]. And the recent research shows that the quantum phase transition of Dicke model with inclusion of the $A^2$-term is forbidden due to the so-called Thomas-Reiche-Kuhn (TRK) sum rule [20, 21]. It can be seen also from the Eq. (8) that turning the ratio between $4\lambda^4$ and $\lambda^1$ across 1, determines the occurrence of the QPT, but however the parameter values of $\lambda$ and $\varepsilon$ are not arbitrary and are restricted by the TRK sum rule for the atom [11].

3. GENERALIZED DICKE MODEL WITH DIPOLE-DIPOLE INTERACTION

Next we focus on the generalized Dicke model with dipole-dipole interaction, which has been studied in the Ref. [22] based on an algebraic Bethe ansatz. The Hamiltonian reads

$$H^{(2)} = \omega a^\dagger a + \omega_0 J_z + \frac{\lambda}{\sqrt{N}} (J_- + J_+) \left( a^\dagger + a \right) + \frac{\Omega}{N} J_z J_-$$

where $\Omega$ describes the direct atom-atom interaction and the prefactors $1/N$ inserted is in the same purpose as previous $1/\sqrt{N}$.

In the thermodynamic limit, $N \gg 1$, the quantum fluctuations are small and may be treated in a linearized approach. Following [9, 10] we perform the Holstein-Primakoff transformation of the angular momentum operators [23]. Collective atomic operators $J_\pm$ and $J_z$ are defined as

$$J_+ = b^\dagger \sqrt{N - b^\dagger b}$$
$$J_- = \sqrt{N - b^\dagger b} b$$
$$J_z = b^\dagger b - N/2$$

where the induced operators $b$ and $b^\dagger$ satisfy Bose commutator relation $[b, b^\dagger] = 1$. Then we introduce two shifting boson operators $b^\dagger = d^\dagger - \sqrt{N} \alpha$ and $a^\dagger = e^\dagger + \sqrt{N} \beta$ with properly scaled auxiliary parameters $\alpha$ and $\beta$ to describe the collective behaviors of both the atoms and the photons [10, 24]. By means of the boson expansion approach, we expand the $H^{(2)}$ with respect to the new operators $e^\dagger$ and $d^\dagger$ as power series of $1/\sqrt{N}$. Substituting these expressions into Eq. (11) the scaled ground-state energy is given by

$$\frac{E_g(\alpha, \beta)}{N} = \omega \beta^2 - 4\lambda \alpha \beta \sqrt{1 - \alpha^2} + (\omega_0 + \Omega) \alpha^2 - \Omega \alpha^4 - \frac{\omega_0}{2}$$
In the above derivations the following approximation
\[
1 - \left[ d_i^+ d_i - \frac{\sqrt{N} \alpha (d_i^+ d_i)}{k N} \right] = 1 + \frac{\alpha (d_i^+ d_i)}{2 k \sqrt{N}} - \frac{1}{2 k \sqrt{N}} = \left[ d_i^+ d_i + \frac{\alpha^2 (d_i^+ d_i)^2}{4 k} \right]
\]
was used [15], with \( k = 1 - \alpha^2 \). The critical points can be determined from the equilibrium conditions which leads to two coupled equations with respect to \( \alpha \) and \( \beta \)
\[
\begin{align*}
\alpha (\Omega + \omega_0) - 2 \lambda \beta \sqrt{k} - 2 \alpha^3 \Omega & = 0 \\
\omega \beta - 2 \alpha \lambda \sqrt{k} & = 0
\end{align*}
\]
(14)

After a little algebra we can obtain the following self-consistency equations
\[
\alpha = \begin{cases} 
0, & \lambda < \lambda_c \\
\eta, & \lambda > \lambda_c
\end{cases}
\]
(15)
\[
\beta = \begin{cases} 
0, & \lambda < \lambda_c \\
\frac{2 \lambda \eta \sqrt{1 - \eta^2}}{\omega}, & \lambda > \lambda_c
\end{cases}
\]
(16)

where \( \eta = \sqrt{4 \lambda^2 - \omega (\Omega + \omega_0)(8 \lambda^2 - 2 \Omega \omega)} \).

One can see from the Eqs. (15,16) that in the normal phase (\( \lambda < \lambda_c \)) the system is unexcited, whereas above the transition point both the field and the atomic ensemble acquire macroscopic excitations. The QPT for the generalized DM with DDI occurs at the critical point \( \lambda_c \) given by
\[
\lambda_c = \sqrt{\omega_0 \omega_0 / 2}
\]
(17)

with \( \tilde{\omega}_0 = \omega_0 + \Omega \), which agrees with the previous results [15,16]. The Eq. (17) is of great significance for the advanced realization of the Dicke superradiation phase transition because \( \Omega \) may be negative for spin-spin interaction between solid qubits. And the corresponding scaled ground state energy reads
\[
\frac{E_g}{N} = \begin{cases} 
- \frac{\omega_0}{\lambda}, & \lambda < \lambda_c \\
-4 \lambda^2 \eta^2 (1 - \eta^2) / \omega - \eta^2 \Omega + \eta^2 (\Omega + \omega_0) - \omega_0 / 2, & \lambda > \lambda_c
\end{cases}
\]
(18)

It can be seen that the ground state energy is independent of the atom-field coupling parameter \( \lambda \) and DDI parameter \( \Omega \) in normal phase which means that the atom-field and atom-atom interactions are absent, whereas in superradiation phase the ground state energy is governed by the parameters \( \lambda \) and \( \Omega \) and obviously DDI has an influence on the ground state energy. The scaled ground-state energy \( E_g / N \) and its second-order derivative as a function of coupling constant \( \lambda \) are plotted in Fig. 2 with \( \omega = \omega_0 = 1 \) (on scaled resonance) and \( \Omega = 0.2 \), from which it can be seen that the system undergoes a second-order phase transition. With comparison to the non-RWA critical value \( \sqrt{\omega_0 \omega_0 / 2} \) for the standard DM where \( \Omega = 0 \), one can see that
the DDI induces a correction of the critical value of QPT and amounts to a shift of the atomic level splitting.

4. CONCLUSION

In conclusion, we derived the critical behavior for two generalized Dicke models with either the $A^2$-term or with dipole-dipole interaction, by means of spin-coherent-state representation and Holstein-Primakoff representation, respectively. It is shown that both the $A^2$-term and dipole-dipole interaction induce a shift of radiation field frequency and atomic level splitting, respectively, resulting in a corrected position of the critical point.

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