A NOTE ON CLASSIFICATION OF TELEPARALLEL CONFORMAL VECTOR FIELDS IN BIANCHI TYPE I SPACE-TIMES IN THE TELEPARALLEL THEORY OF GRAVITATION

GHULAM SHABBIR¹, HINA KHAN
Faculty of Engineering Sciences, GIK Institute of Engineering Sciences and Technology, Topi, Swabi, Khyber Pakhtunkhwa, Pakistan. ¹E-mail: shabbir@giki.edu.pk
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A study of teleparallel conformal vector fields in Bianchi type I space-times in the teleparallel theory of gravitation is given by using the direct integration technique. From the above study it turns out that the dimensions of teleparallel conformal vector fields are 8, 9, 10 or 11. The case when the above space-times become FRW $k = 0$ model it admits eleven teleparallel conformal vector fields. This classification also covers the timelike version of plane symmetric space-times.

Key words: Weitzenböck geometry, Torsion, Teleparallel conformal vector fields.
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1. INTRODUCTION

Symmetries play an important role in the theory of general relativity. Over the past number of years the researchers are studying different kind of symmetries such as Killing, homothetic, conformal, affine curvature Ricci and projective vector fields. We impose these symmetry restrictions on the space-time metric to solve the gravitational field equations, which are highly non-linear. Laws of conservation of the matter in the space-time can be studied with the help of these symmetry restrictions. These symmetry restrictions not only give us the laws of conservation [1] but also provide some geometrical features and physical information about the space-time. For example, in general relativity self-similarity solutions are extensively used for cosmological perturbations, star formation, gravitational collapse, primordial black holes, cosmological voids and cosmic censorship [2]. One can see there are other theories of gravitation where researchers are working to find some reasonably good results for gravitational behaviour. Teleparallel is one such theory where gravitation is attributed only to the torsion of the space-time by considering the space-time curvature as zero and this torsion plays the role of a force [3]. Teleparallel theory of gravity is based on
Weitzenböck geometry [4]. In [5] the authors introduced the teleparallel version of the Lie derivative for Killing vector fields and used those equations to find the teleparallel Killing vector fields in Einstein universe. In [6-13] the authors made further progress to classify different space-times according to their teleparallel Killing vector fields. From the above study they showed that some time teleparallel theory gives more conservation laws as compared with the theory of general relativity.

In [14-16] the authors give classification of space-times according to their proper teleparallel homothetic vector fields and showed that they give one extra conservation law. Keeping in view the importance of this teleparallel theory we extend our study to teleparallel conformal vector fields [17] in which we find that very special class of cylindrically static space-times admit conformal vector fields. It will be interesting to further explore conformal vector fields in teleparallel theory of gravity and to see if the results of teleparallel and general relativity coincide. The current study, in Bianchi type I space-times, will not only help to understand the geometrical and physical properties of the space-time but also to find the effect of torsion on the laws of gravitation. These results may give us interesting information about the compatibility of both the theories. The procedure for finding Weitzenböck connections and torsion components are given in detail in [13].

Teleparallel Killing equation is defined as [5]

\[ L^T_X g_{\mu\nu} = g_{\mu\nu,\rho} X^\rho + g_{\mu\rho} X^\nu,\rho + g_{\nu\rho} X^\mu,\rho + X^\rho (g_{\mu\nu,\rho} + g_{\nu\rho,\mu}) = 0, \]

where \( L^T_X \) denotes the teleparallel Lie derivative with respect to the vector field \( X \). A vector \( X \) is said to be teleparallel conformal vector fields if it satisfies the relation

\[ L^T_X g_{\mu\nu} = 2\phi g_{\mu\nu}, \]

where \( \phi \) is a function on \( M \). If \( \phi \) becomes constant on \( M \), then \( X \) is called teleparallel homothetic [14-16] while if \( \phi = 0 \) it becomes teleparallel Killing vector fields.

2. MAIN RESULTS

Consider Bianchi type I space-times in usual coordinates \((t, x, y, z)\) (labeled by \((x^0, x^1, x^2, x^3)\), respectively) with the line element [18]

\[ ds^2 = -dt^2 + e^{2A(t)}dx^2 + e^{2B(t)}dy^2 + e^{2C(t)}dz^2, \]
where \( A, B \) and \( C \) are functions of \( t \) only. The \( \gamma \)-tetrad components and its inverse can be obtained as

\[
S^\gamma_\mu = \text{diag}(-1, e^{A(t)}, e^{B(t)}, e^{C(t)}), \quad S^\mu_\gamma = \text{diag}(-1, e^{-A(t)}, e^{-B(t)}, e^{-C(t)}).
\]

The non-vanishing Weitzenböck connections are obtained as

\[
\Gamma_{10}^1 = \dot{A}, \quad \Gamma_{20}^2 = \dot{B}, \quad \Gamma_{30}^3 = \dot{C}.
\]

Thus the non vanishing torsion components are

\[
T_{01}^1 = -T_{10}^1 = \dot{A}, \quad T_{02}^2 = -T_{20}^2 = \dot{B}, \quad T_{03}^3 = -T_{30}^3 = \dot{C},
\]

where dot denotes the derivative with respect to \( t \). A vector field \( X \) is said to be teleparallel conformal vector field if it satisfies equation (2). Now using (3) and (6) in (2) we get the teleparallel conformal equations as follows

\[
X^0_0 = \phi, \quad X^1_1 = \phi, \quad X^2_2 = \phi, \quad X^3_3 = \phi,
\]

\[
e^{2A(t)} X^1,2 + e^{2B(t)} X^2,1 = 0,
\]

\[
e^{2A(t)} X^1,3 + e^{2C(t)} X^3,1 = 0,
\]

\[
e^{2B(t)} X^2,3 + e^{2C(t)} X^3,2 = 0,
\]

\[
X^0,3 - e^{2A(t)} X^1,0 - e^{2A(t)} \dot{A} X^1 = 0,
\]

\[
X^0,2 - e^{2B(t)} X^2,0 - e^{2B(t)} \dot{B} X^2 = 0,
\]

\[
X^0,1 - e^{2C(t)} X^3,0 - e^{2C(t)} \dot{C} X^3 = 0.
\]

Now integrating equation (7), we get

\[
X^0 = \int \phi dt + P^1(x, y, z), \quad X^1 = \int \phi dx + P^2(t, y, z),
\]

\[
X^2 = \int \phi dy + P^3(t, x, z), \quad X^3 = \int \phi dz + P^4(t, x, y),
\]

where \( P^1(x, y, z), P^2(t, y, z), P^3(t, x, z) \) and \( P^4(t, x, y) \) are functions of integration which are to be determined. To avoid lengthy details here we shall write only the results which are

**Case N:** In this case we have \( A \neq B, A \neq C \) and \( B \neq C \). The space-time is given in equation (3). Solution of equations (7) to (13) is given by
\[ X^0 = \int \phi(t) dt + c_1 \frac{x^2}{2} + c_2 \frac{y^2}{2} + c_3 \frac{z^2}{2} + c_4 x + c_5 y + c_6 z + c_7, \]
\[ X^1 = x \phi + c_8 e^{-4 \int e^{-\phi} dt} + c_9 e^{-d}, \]
\[ X^2 = y \phi + c_5 e^{-b \int e^{-\phi} dt} + c_5 e^{-b}, \]
\[ X^3 = z \phi + c_6 e^{-c \int e^{-\phi} dt} + c_{10} e^{-c}. \] (15)

The conformal factor is \( \phi = \phi(t) \) and the unknown metric coefficients are
\[ e^A = \left( \frac{c_1}{\phi} \int e^{-\phi} dt + \frac{c_{11}}{\phi} \right), \quad e^B = \left( \frac{c_2}{\phi} \int e^{-\phi} dt + \frac{c_{12}}{\phi} \right), \quad e^C = \left( \frac{c_3}{\phi} \int e^{-\phi} dt + \frac{c_{13}}{\phi} \right), \]
where \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13} \in R. \) Here, the above space-times (3) admit eight linear independent teleparallel conformal vector fields.

**Case O:** In this case there exist the following three possibilities which are
(a1): \( A = A(t), \quad B = B(t) \) and \( C = \text{constant} \)
(a2): \( A = A(t), \quad B = \text{constant} \) and \( C = C(t) \)
(a3): \( A = \text{constant}, \quad B = B(t) \) and \( C = C(t) \)

In (a1) the space-time (3) can, after a suitable rescaling of \( z \), be written in the form
\[ ds^2 = -dt^2 + e^{2A(t)} dx^2 + e^{2B(t)} dy^2 + dz^2. \] (16)

Teleparallel conformal vector fields in this case are
\[ X^0 = (c_1 \frac{t^2}{2} + c_3 t) + c_1 \frac{x^2}{2} + c_2 \frac{y^2}{2} + c_3 \frac{z^2}{2} + c_4 x + c_5 y + c_6 z + c_7, \]
\[ X^1 = x(c_3 t + c_{11}) + c_4 e^{-4 \int e^{-\phi} dt} + c_5 e^{-d}, \]
\[ X^2 = y(c_4 t + c_{12}) + c_5 e^{-b \int e^{-\phi} dt} + c_5 e^{-b}, \]
\[ X^3 = z(c_5 t + c_{13}) + c_6 e^{-c \int e^{-\phi} dt} + c_{10} e^{-c}. \] (17)

The conformal factor is \( \phi = c_3 t + c_{13} \) and the unknown metric coefficients are
\[ e^A = \left( \frac{c_1}{\phi} \int e^{-\phi} dt + \frac{c_{11}}{\phi} \right) \quad \text{and} \quad e^B = \left( \frac{c_2}{\phi} \int e^{-\phi} dt + \frac{c_{12}}{\phi} \right), \]
where \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13} \in R. \) In this case the above space-times (16) admit nine linear independent teleparallel conformal vector fields. Cases (a2) and (a3) are exactly the same.

**Case P:** In this case the following three possibilities exist, which are
(b1): \( A = A(t) \) and \( B(t) = C(t) \)
(b2): \( B = B(t) \) and \( A(t) = C(t) \)
(b3): \( C = C(t) \) and \( A(t) = B(t) \)
In (b1) the space-time (3) takes the form
\[ ds^2 = -dt^2 + e^{2A(t)}(dx^2 + dy^2 + dz^2). \] (18)

Teleparallel conformal vector fields in this case are
\[ X^0 = \int \phi(t) dt + c_1 \frac{x^2}{2} + c_2 \frac{y^2}{2} + c_3 \frac{z^2}{2} + c_4 x + c_5 y + c_6 z + c_7, \]
\[ X^1 = x\phi + c_4 e^{-A} \int e^{-A} dt + c_4 e^{-A}, \] (19)
\[ X^2 = y\phi + c_4 e^{-B} \int e^{-B} dt - c_3 y e^{-B} + c_9 e^{-B}, \]
\[ X^3 = z\phi + c_4 e^{-B} \int e^{-B} dt + c_3 z e^{-B} + c_{10} e^{-B}. \]

The conformal factor is \( \phi = \phi(t) \) and the unknown metric coefficients are
\[ e^A = \left( \frac{c_1}{\phi} \int e^{-A} dt + \frac{c_{11}}{\phi} \right) \]
\[ e^B = \left( \frac{c_2}{\phi} \int e^{-B} dt + \frac{c_{12}}{\phi} \right), \]
where \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12} \in \mathbb{R} \). Here, the above space-times (18) admit nine linear independent teleparallel conformal vector fields. Other cases (b2) and (b3) are exactly the same.

**Case Q:** In this case there exist the following three possibilities, which are
(d1): \( A = \) constant and \( B(t) = C(t) \).
(d2): \( B = \) constant and \( A(t) = C(t) \).
(d3): \( C = \) constant and \( A(t) = B(t) \).

In (d1) the space-time (3) can, after a suitable rescaling of \( x \), be written in the form
\[ ds^2 = -dt^2 + dx^2 + e^{2A(t)}(dy^2 + dz^2). \] (20)

Teleparallel conformal vector fields in this case are
\[ X^0 = (c_2 \frac{t^2}{2} + tc_{11}) + c_3 \frac{x^2}{2} + c_5 \frac{y^2}{2} + c_7 x + c_8 y + c_6 z + c_7, \]
\[ X^1 = x(c_2 t + c_{11}) + c_4 t + c_8, \] (21)
\[ X^2 = y(c_2 t + c_{11}) + c_5 e^{-B} \int e^{-B} dt + c_3 z e^{-B} + c_9 e^{-B}, \]
\[ X^3 = z(c_2 t + c_{11}) + c_6 e^{-B} \int e^{-B} dt - c_3 y e^{-B} + c_{10} e^{-B}. \]

The conformal factor is \( \phi = c_2 t + c_{11} \) and the unknown metric coefficient is
\[ e^B = \left( \frac{c_1}{\phi} \int e^{-B} dt + \frac{c_{12}}{\phi} \right), \]
where \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12} \in \mathbb{R} \).
Here, the above space-times (20) admit ten linear independent teleparallel conformal vector fields. Other cases (d2) and (d3) are exactly the same.

**Case R:** In this case there exist the following three possibilities, which are

(e1): \( A = A(t) \) and \( B = C = \text{constant} \)

(e2): \( B = B(t) \) and \( A = C = \text{constant} \)

(e3): \( C = C(t) \) and \( A = B = \text{constant} \)

In (e1) the space-time (3) can, after a suitable rescaling of \( y \) and \( z \), be written in the form

\[
\text{ds}^2 = -dt^2 + e^{2A(t)}dx^2 + dy^2 + dz^2.
\]  

(22)

Teleparallel conformal vector fields in this case are

\[
X^0 = (c_2 \frac{t^2}{2} + tC_{11}) + c_1 \frac{x^2}{2} + c_3 \frac{y^2}{2} + c_4x + c_5y + c_6z + c_7, \\
X^1 = x(c_2t + c_{11}) + c_4e^{-A} \int e^{-A} dt + c_8e^{-A}, \\
X^2 = y(c_2t + c_{11}) + c_5 t + c_4z + c_9, \\
X^3 = z(c_2t + c_{11}) + c_6 t - c_3y + c_{10}.
\]  

(23)

The conformal factor is \( \phi = c_2t + c_{11} \) and the unknown metric coefficient is

\[
e^A = \left( \frac{c_1}{\phi} \int e^{-A} dt + \frac{c_{12}}{\phi} \right), \quad \text{where} \quad c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12} \in R.
\]

Here, the above space-times (22) admit ten linear independent teleparallel conformal vector fields. Other cases (e2) and (e3) are exactly the same.

**Case S:** In this case we have \( A(t) = B(t) = C(t) \). The line element takes the form

\[
\text{ds}^2 = -dt^2 + e^{2A(t)}(dx^2 + dy^2 + dz^2).
\]  

(24)

The above space-times (24) become FRW \( k = 0 \) model. Teleparallel conformal vector fields in this case are

\[
X^0 = \int \phi(t) dt + c_1 \left( \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} \right) + c_4x + c_5y + c_6z + c_7, \\
X^1 = x\phi + c_4e^{-A} \int e^{-A} dt + c_2 y e^{-A} + c_3 z e^{-A} + c_8 e^{-A}, \\
X^2 = y \phi + c_4 e^{-A} \int e^{-A} dt - c_2 y e^{-A} + c_12 z e^{-A} + c_9 e^{-A}, \\
X^3 = z \phi + c_4 e^{-A} \int e^{-A} dt - c_3 y e^{-A} - c_12 y e^{-A} + c_{10} e^{-A}.
\]  

(25)

The conformal factor is \( \phi = \phi(t) \) and the unknown metric coefficient is

\[
e^A = \left( \frac{c_1}{\phi} \int e^{-A} dt + \frac{c_{12}}{\phi} \right), \quad \text{where} \quad c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12} \in R.
\]
Here, the above space-times (24) admit eleven linear independent teleparallel conformal vector fields.

3. CONCLUSION

In this paper we classified the teleparallel conformal vector fields for Bianchi type I space-times. From the above study it is shown that there exist six major cases which are N, O, P, Q, R and S which admit 8, 9, 9, 10, 10 and 11 teleparallel conformal vector fields, respectively. It also turns out that the teleparallel conformal vector fields are multiple of some specific functions of $t$. These functions appear in teleparallel conformal vector fields because of the non-vanishing torsion components. It is important to mention here that in the case S the space-time (24) becomes FRW $k = 0$ model and admits eleven linear independent teleparallel conformal vector fields. We hope this paper will help to understand teleparallel conformal vector fields in both the theories which in turn will deepen our understanding of the space-time structure and its physical behavior in this alternate teleparallel theory of gravitation.

REFERENCES